Proving DFA to CFG Construction

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1 Introduction

How does one prove that all DFA's recognize CFL's. Here are two ways. Let M be a DFA.

1. Since all DFA's are PDA's, M is a PDA. For all PDA's M there exists CFL G such that $L(M) = L(G)$. The drawback of this proof is that it requires PDA-to-CFG theorem.
2. For all DFA's M there exists a regular expression α such that $L(M) = L(\alpha)$. By induction on the formation of a regular expression one can easily show that, for all regular expressions α, $L(\alpha)$ is a CFL. The drawback of this proof is that it requires the DFA-to-Reg Expression theorem.

Can one transform a DFA into a CFG directly? Yes. We give the proof in the next section. It resembles the proof that if M is DFA then there is a regular expression for α such that $L(M) = L(\alpha)$. (The $R(i, j, k)$ construction.)

2 The Direct Proof

Theorem If M is a DFA then there exists CFG G such that $L(M) = L(G)$.
Proof:
Let $M = (Q, \Sigma, \delta, s, F)$. We can assume $Q = \{1, \ldots, n\}$ and that $s = 1$.
We want to form CFG $G = (N, \Sigma, P, S)$ such that $L(G) = L(M)$.

- $N = \{[i, j, k] : (i, j, k) \in Q \times Q \times \{0, \ldots, n\}\} \cup \{S\} \cup \{[k, k, k-1, *] : k \in \{2, \ldots, n\}\}$.

  Our intention is that

  $[i, j, k] \Rightarrow \{w \in \Sigma^* : \overline{\delta}(i, w) = j$ and the path only uses states $\leq k\}$.
- S is the start state.
- We define P the set of productions.
  - For every $f \in F$ we have the production $S \to (1, f, n)$.

- *Generating base productions.* For every $i, j \in Q$,
  $[i, j, 0] \rightarrow \{\sigma \in \Sigma : \overline{\delta}(i, \sigma) = j\}$
  $[i, i, 0] \rightarrow e$
- *Generating the rest of the productions.* For every $i, j \in Q$, and every $1 \leq k \leq n$:
  $[i, j, k] \rightarrow [i, j, k-1]$

  $[i, j, k] \rightarrow [i, k, k-1][k, k, k-1, *][k, j, k-1]$

  $[k, k, k-1, *] \rightarrow [k, k, k-1][k, k, k-1, *]$

  $[k, k, k-1, *] \rightarrow e$

The reader can show by induction on k that, for all k, i, j,
$[i, j, k] \Rightarrow \{w \in \Sigma^* : \overline{\delta}(i, w) = j$ and the path only uses states $\leq k\}$.