An Open Problem about CFG's and decidability and
An NP-hardness result
An Exposition by Bill Gasarch

1 Introduction

Def 1.1 If w is a string then $n_a(w)$ is the number of a's in w.

2 Counting Descriptions and CFG's

Def 2.1 A *counting Description* is a boolean combination of linear equations and inequalities involving $n_sigma(w)$. For example

$$(n_a(w) \le 2n_b(w) + 3) \wedge \neg(n_c(w) = n_b(w)).$$

If E is a counting description then

$$L(E) = \{w : E(w) \text{ is true .}\}$$

We believe the complexity of the following problems (even if they are decidable) is open:

1. Given a counting description E, is $L(E)$ regular?
2. Given a counting description E, is $L(E)$ context free?

We can vary these problems in the following ways (and combinations of these ways).

1. Do not allow negation.
2. Do not allow intersection.
3. Do not allow inequalities.
4. Do not allow additive constants.
5. Do not allow additive constants.
6. Do not allow multiplicative constants.
7. Allow other types of equations, for example $n_a(w) = n_b(w)^2$. (If allow any polys then undecidably by Hilbert's Tenth problem.)

3 An NP-Hardness Result

.

The following is due to Richard Beigel.

Theorem 3.1 *There is a polytime reduction that will, given a CNF formula ϕ produce a description E such that the following happens.*

- *If ϕ is satisfiable then $L(E)$ is not context free.*
- *If ϕ is not satisfiable then $L(E) = \emptyset$.*

Hence all of the problems above are NP-hard.

Proof:

Given a CNF formula $\phi(x_1, \ldots, x_n)$ we let $\Sigma = \{a, b, c, x_1, x_2, \ldots, x_n\}$.

Our counting description E is the AND of the following.

1. $n_a(w) = n_b(w) \wedge n_a(w) = n_c(w) \wedge n_b(w) = n_c(w)$.
2. For each i $0 \le n_{x_i}(w) \le 1$.
3. For each clause C do the following. For every positive literal x in the clause we have $n_x(w)$. For each negative literal $\neg x$ we have $1 - n_x(w)$. We have the condition that the sum of these is ≥ 1.

Let $L = \{w : n_a(w) = n_b(w) = n_c(w)\}$.

If ϕ is satisfiable then $L(E) \cup a^*b^*c^* = L$ and hence $L(E)$ is NOT context free.

If ϕ is NOT satisfiable then $L(E) = \emptyset$.

∎