Standard and NonStandard Dice: An Exposition

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If you roll two standard 6-sided dice then

1. 2: (1,1). ONE way. Prob $\frac{1}{36}$.
2. 3: (1,2), (2,1). TWO ways. Prob $\frac{1}{18}$.
3. 4: (1,3), (2,2), (3,1). THREE ways. Prob $\frac{1}{12}$.
4. 5: (1,4), (2,3), (3,2), (4,1). FOUR ways. Prob $\frac{1}{9}$.
5. 6: (1,5), (2,4), (3,3), (4,2), (5,1) FIVE ways. Prob $\frac{5}{36}$.
6. 7: (1,6), (2,5), (3,4), (4,3), (5,2), (6,1) SIX ways. Prob $\frac{1}{6}$.
7. 8: (2,6), (3,5), (4,4), (5,3), (6,2) FIVE ways. Prob $\frac{5}{36}$.
8. 9: (3,6), (4,5), (5,4), (6,2) FOUR ways. Prob $\frac{1}{9}$.
9. 10: (4,6), (5,5), (5,6) THREE ways. Prob $\frac{1}{12}$.
10. 11: (5,6), (6,5) TWO ways. Prob $\frac{1}{18}$.
11. 12: (6,6) ONE way. Prob $\frac{1}{36}$.
Let Polynomials Do The Work For You!

\[(x^6 + x^5 + x^4 + x^3 + x^2 + x)(x^6 + x^5 + x^4 + x^3 + x^2 + x)\]
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\[(x^6 + x^5 + x^4 + x^3 + x^2 + x)(x^6 + x^5 + x^4 + x^3 + x^2 + x)\]

Look at coefficient of \(x^6\)
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Look at coefficient of \(x^6\)

\[x^1x^5 + x^2x^4 + x^3x^3 + x^4x^2 + x^5x^1 = 5x^6 = (\text{Number of ways to get 6})x^6\]
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Look at coefficient of \(x^6\)

\[x^1x^5 + x^2x^4 + x^3x^3 + x^4x^2 + x^5x^1 = 5x^6 = (\text{Number of ways to get 6})x^6\]

Coefficient of \(x^n\) is number of ways to get \(n\).
Example of Non-Standard Labelings

What is we label the dice \((1, 2, 2, 2, 5, 5)\) and \((1, 3, 3, 3, 3, 7)\)?
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What is we label the dice (1, 2, 2, 2, 5, 5) and (1, 3, 3, 3, 3, 7)?

\[(2x^5 + 3x^2 + x)(x^7 + 4x^3 + x) = 2x^{12} + 3x^9 + 9x^8 + 2x^6 + 12x^5 + 4x^4 + 3x^3 + x^2\]
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What is we label the dice \((1, 2, 2, 2, 5, 5)\) and \((1, 3, 3, 3, 3, 7)\)?

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1. 12: TWO ways. Prob \(\frac{1}{18}\).
2. 9: THREE ways. Prob \(\frac{1}{6}\).
3. 8: NINE ways. Prob \(\frac{1}{4}\).
4. 6: TWO ways. Prob \(\frac{1}{18}\).
5. 5: TWELVE ways. Prob \(\frac{1}{3}\).
6. 4: FOUR ways. Prob \(\frac{1}{9}\).
7. 3: THREE ways. Prob \(\frac{1}{12}\).
8. 2: ONE ways. Prob \(\frac{1}{36}\).
Is there a Non-Standard Labeling That...

**Question** Is there a nonstandard labeling of dice that gives the same probabilities as the standard dice?
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**Question Phrased In Terms of Polynomials** Does there exist $a_1 \geq \cdots \geq a_6$ and $b_1 \geq \cdots \geq b_6$ such that

$$\left(x^{a_1} + x^{a_2} + x^{a_3} + x^{a_4} + x^{a_5} + x^{a_6}\right)\left(x^{b_1} + x^{b_2} + x^{b_3} + x^{b_4} + x^{b_5} + x^{b_6}\right) = \left(x^6 + x^5 + x^4 + x^3 + x^2 + x\right)^2.$$
Question Is there a nonstandard labeling of dice that gives the same probabilities as the standard dice?

Question Phrased In Terms of Polynomials Does there exist $a_1 \geq \cdots \geq a_6$ and $b_1 \geq \cdots \geq b_6$ such that

$$(x^{a_1} + x^{a_2} + x^{a_3} + x^{a_4} + x^{a_5} + x^{a_6})(x^{b_1} + x^{b_2} + x^{b_3} + x^{b_4} + x^{b_5} + x^{b_6}) =$$

$$(x^6 + x^5 + x^4 + x^3 + x^2 + x)^2.$$
Is there a Non-Standard Labeling That... Cont.

\[(x^{a_1} + x^{a_2} + x^{a_3} + x^{a_4} + x^{a_5} + x^{a_6})(x^{b_1} + x^{b_2} + x^{b_3} + x^{b_4} + x^{b_5} + x^{b_6}) =\]

\[(x^6 + x^5 + x^4 + x^3 + x^2 + x)^2 = x^2(x^5 + x^4 + x^3 + x^2 + x + 1)^2 =\]

\[x^2(x + 1)^2(x^2 - x + 1)^2(x^2 + x + 1)^2.\]
Is there a Non-Standard Labeling That... Cont.

\[(x^{a_1} + x^{a_2} + x^{a_3} + x^{a_4} + x^{a_5} + x^{a_6})(x^{b_1} + x^{b_2} + x^{b_3} + x^{b_4} + x^{b_5} + x^{b_6}) = \]

\[(x^6 + x^5 + x^4 + x^3 + x^2 + x)^2 = x^2(x^5 + x^4 + x^3 + x^2 + x + 1)^2 = \]

\[x^2(x + 1)^2(x^2 - x + 1)^2(x^2 + x + 1)^2.\]

What properties do the polys we are looking for have?

1. \(a_6 = 1\) and \(b_6 = 1\) since otherwise cannot get a 2. So both poly’s have a factor of \(x\).

2. \((1^{a_1} + 1^{a_2} + x^{a_3} + 1^{a_4} + 1^{a_5} + 1^{a_6}) = 6.\) So if \(f(x)\) is a factor need \(f(1) = 6.\)
Is there a Non-Standard Labeling That... Cont.

\[ x^2(x + 1)^2(x^2 - x + 1)^2(x^2 + x + 1)^2 = \]

\[ x(x+1)^a(x^2-x+1)^b(x^2+x+1)^c \times x(x+1)^2-a(x^2-x+1)^2-b(x^2+x+1)^2-c. \]
Is there a Non-Standard Labeling That... Cont.

\[ x^2(x + 1)^2(x^2 - x + 1)^2(x^2 + x + 1)^2 = \]

\[ x(x+1)^a(x^2-x+1)^b(x^2+x+1)^c = x(x+1)^{2-a}(x^2-x+1)^{2-b}(x^2+x+1)^{2-c}. \]

Since \( f(1) = 6 \) and \( g(1) = 6 \) we have conditions
\[ 1 \times 2^a \times 1^b \times 3^c = 6 \text{ and } 1 \times 2^{2-a} \times 1^{2-b} \times 3^{2-c} = 6. \]
So
\[ a = 1 \quad b \in \{0, 1, 2\} \quad c = 1. \]
Is there a Non-Standard Labeling That... Cont.

\[ x^2(x + 1)^2(x^2 - x + 1)^2(x^2 + x + 1)^2 = \]

\[ x(x+1)^a(x^2-x+1)^b(x^2+x+1)^c \times x(x+1)^{2-a}(x^2-x+1)^{2-b}(x^2+x+1)^{2-c}. \]

Since \( f(1) = 6 \) and \( g(1) = 6 \) we have conditions
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So

\[ a = 1 \quad b \in \{0, 1, 2\} \quad c = 1. \]

\( b = 0 \) and \( b = 2 \) are symmetric so we just do \( b = 0 \) and \( b = 1 \).
The Non-Standard Labeling

**Case** $b = 0$: Then the polynomials for the dice are
\[ x(x + 1)(x^2 + x + 1) = x^4 + 2x^3 + 2x^2 + x. \]
\[ x(x + 1)(x^2 - x + 1)^2(x^2 + x + 1) = x^8 + x^6 + x^5 + x^4 + x^3 + x. \]
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So the dice are $(1, 2, 2, 3, 3, 4)$ and $(1, 3, 4, 5, 6, 8)$.
Great- these are nonstandard dice that give the same probs as standard dice.
The Non-Standard Labeling

Case \( b = 0 \): Then the polynomials for the dice are
\[
x(x + 1)(x^2 + x + 1) = x^4 + 2x^3 + 2x^2 + x.
\]
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x(x + 1)(x^2 - x + 1)^2(x^2 + x + 1) = x^8 + x^6 + x^5 + x^4 + x^3 + x.
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So the dice are \((1, 2, 2, 3, 3, 4)\) and \((1, 3, 4, 5, 6, 8)\).
GREAT- these are nonstandard dice that give the same probs as standard dice.

Case \( b = 1 \): Then the polynomials for the dice are
\[
x(x + 1)(x^2 - x + 1)(x^2 + x + 1) = (x^6 + x^5 + x^4 + x^3 + x^2 + x).
\]
\[
x(x + 1)(x^2 - x + 1)(x^2 + x + 1) = (x^6 + x^5 + x^4 + x^3 + x^2 + x).
\]
So the dice are \((1, 2, 3, 4, 5, 6)\) and \((1, 2, 3, 4, 5, 6)\).
The standard dice.

Upshot there is only ONE pair of nonstandard dice that give the same probabilities as the standard dice. That pair is \((1, 2, 2, 3, 3, 4)\) and \((1, 3, 4, 5, 6, 8)\).