When Does a Number Equal the Sum of the Squares or Cubes of its Digits?
An Exposition and a Call for a More elegant Proof
by William Gasarch

1 Introduction

We will look at theorems of the following form:

**Theorem 1.1** Let $L < U$ and $k$ be in $\mathbb{N}$. For every $L \leq x \leq U$ there are XXX numbers that are the sum of the $k$th powers of their digits.

The theorems we prove are easy and known. We will give several proofs of some of them with an eye towards the question: What is an elegant proof? For the last theorem I have a proof that is not elegant. If you know a more elegant one then please let me know.

For each proof we note how many times we had to actually CHECK a number. This is a crude measure of elegance— the fewer checks the more elegant the proof (compared to other proofs of the same theorem). We try not to *game the system* by having rather elaborate proofs that don’t check that much.

2 When is a 2-Digit Number the Sum of the Squares of its Digits?

We will prove the following theorem several different ways.

**Theorem 2.1** If $11 \leq x \leq 99$ then $x$ is not the sum of the squares of its digits.

2.1 Proof by Enumeration

One can prove this theorem by listing all numbers between 11 and 99 and seeing that none of them work. One could write a program to produce this proof. We omit this proof.

**Number of Checks:** 89.

**PROS:** You get the answer easily. If you wrote a program then you can adjust it to find the answers. to many more problems of this type (e.g., How many 3-digits numbers are the sum of the cubes of their digits) or variants of this problem (e.g., How many 2-digit numbers are within 1 of the sum of their digits? Which ones are they?)

**CONS:** You get no insights.

2.2 Proof by Looking at each 10-number interval

**Proof:**

**Case 1:** $x = 10a$ where $1 \leq a \leq 9$ (so $x$ ends in a 0).

\[
\begin{align*}
    a^2 &= 10a \\
    a &= 10.
\end{align*}
\]
So this cannot happen. Note we did no checks.

**Case 2:** $11 \leq x \leq 19$. Since $11, 12, 13 > 1^2 + 2^2 > 1^2 + 1^1$, none of $11, 12, 13$ work. Since $15, 16, 17, 18, 19 < 1^2 + 5^2 < 2^2 + 2^2 < 1^2 + 8^2 < 1^2 + 9^2$, none of $15, 16, 17, 18, 19$ work. The only number left is $14$ which does not work since $14 \neq 1^2 + 4^2$. Note we did one check.

**Case 3:** Look at $21 \leq x \leq 29$. $21, 22, 23, 24 > 2^2 + 4^2$. $26, 27, 28, 29 < 2^2 + 6^2$. $25$ does not work since $25 \neq 2^2 + 5^2$. Note that we did one check.

**Case 4:** $31 \leq x \leq 39$. $31, 32, 33, 34 > 3^2 + 4^2$. $36, 37, 38, 39 < 3^2 + 6^2$. $35 \neq 3^2 + 5^2$. Note that we did zero checks.

**Case 5:** $41 \leq x \leq 49$. $41, 42, 43, 44 > 4^2 + 4^2$. $46, 47, 48, 49 < 4^2 + 6^2$. $45 \neq 4^2 + 5^2$. Note that we did one check.

**Case 6:** $51 \leq x \leq 59$. $51, 52, 53, 54, 55 > 5^2 + 5^2$. $56, 57, 58, 59 < 5^2 + 9^2$. Note that we did zero checks.

**Case 7:** $61 \leq x \leq 69$. $61, 62, 63, 64 > 6^2 + 4^2$. $66, 67, 68, 69 < 6^2 + 6^2$. $65 \neq 6^2 + 5^2$. Note that we did one check.

**Case 8:** $71 \leq x \leq 79$. $71, 72, 73, 74 > 7^2 + 4^2$. $76, 77, 78, 79 < 7^2 + 6^2$. $75 \neq 7^2 + 5^2$. Note that we did one check.

**Case 9:** $81 \leq x \leq 89$. $81, 82, 83, 84 > 8^2 + 4^2$. $86, 87, 88, 89 < 8^2 + 6^2$. $85 \neq 8^2 + 5^2$. Note that we did one check.

**Case 9:** $91 \leq x \leq 99$. $91, 92 > 9^2 + 2^2 95, 96, 97, 98, 99 < 9^2 + 5^2$. $93 \neq 9^2 + 3^2$, $94 \neq 9^2 + 4^2$. Note that we did two checks.

**Number of Checks:** 8

**PRO:** Only ten cases. The techniques used may be helpful for other problems.

**CON:** Only ten cases? If you do other problems with way the number of cases may get rather large.

### 2.3 Proof Using Mod 2, Mod 4, and Mod 10

**Proof:**

Assume, by way of contradiction, that there exists $1 \leq a \leq 9$ and $0 \leq b \leq 9$ such that

$$10a + b = a^2 + b^2$$

henceforth the main equation.

Take this mod 2 using $x^2 \equiv x \pmod{2}$ and $10 \equiv 0 \pmod{2}$ to obtain

$$a \equiv 0 \pmod{2}.$$
If we take the main equation mod 4, using $a^2 \equiv 0 \pmod{4}$, we get

$$b(b - 1) \equiv 0 \pmod{4}.$$

Hence $b \in \{0, 1, 4, 5, 8, 9\}$.

We now take the main equation mod 10.

$$b \equiv a^2 + b^2 \pmod{10}.$$

$$a = \sqrt{-b(b - 1)} \pmod{10}.$$

We now run through $b \in \{0, 1, 4, 5, 8, 9\}$ and note for which ones $a = \sqrt{-b(b - 1)} \pmod{10}$ exists and is a nonzero even number. Note that the squares mod 10 are 0, 1, 4, 5, 6, 9 so the only nonzero even ones are 4, 6.

<table>
<thead>
<tr>
<th>$b$</th>
<th>$-b(b - 1) \pmod{10}$</th>
<th>$a = \sqrt{-b(b - 1)} \pmod{10}$</th>
<th>comment</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>0</td>
<td>NO GOOD: Need $1 \leq a \leq 9$</td>
</tr>
<tr>
<td>1</td>
<td>0</td>
<td>0</td>
<td>NO GOOD: Need $1 \leq a \leq 9$</td>
</tr>
<tr>
<td>4</td>
<td>8</td>
<td>NONE</td>
<td>NO GOOD: No Square Root</td>
</tr>
<tr>
<td>5</td>
<td>0</td>
<td>0</td>
<td>NO GOOD: Need $1 \leq a \leq 9$</td>
</tr>
<tr>
<td>6</td>
<td>0</td>
<td>0</td>
<td>NO GOOD: Need $1 \leq a \leq 9$</td>
</tr>
<tr>
<td>8</td>
<td>4</td>
<td>2, 8</td>
<td>OH, Will need to check</td>
</tr>
<tr>
<td>9</td>
<td>8</td>
<td>NONE</td>
<td>NO GOOD: No Square Root</td>
</tr>
</tbody>
</table>

So the only candidates for $(a, b)$ are $(2, 4), (8, 4)$.

One can check that these do not satisfy the main equation.

**Number of Checks: 2**

**PRO:** The technique may extends to other problems.

**CAVEAT:** Is this proof really more elegant then the intervals proof?

---

3 When is a $d$-Digit Number the Sum of the Squares of its Digits?

**Theorem 3.1** If $x \geq 2$ then $x$ is not the sum of the squares of its digits.

**Proof:**

**Case 1:** $2 \leq x \leq 9$. Since $x < x^2$ the theorem is true for $2 \leq x \leq 9$.

**Case 2:** $10 \leq x \leq 99$. By Theorem 2.1 our theorem is true for these $x$.

**Case 3:** $1 \leq a \leq 9, 0 \leq b \leq 9, 0 \leq c \leq 9$. Assume

$$x = 100a + 10b + c = a^2 + b^2 + c^2.$$  

The largest $x$ can be is $9^2 + 9^2 + 9^2 < 300$. Hence $a \in \{1, 2\}$.

The largest $x$ can be is $2^2 + 9^2 + 9^2 = 176 < 200$. Hence $a = 1$ and $b \leq 7$.

The largest $x$ can be is $1^2 + 7^2 + 9^2 = 141$. Hence and $b \leq 4$.

The largest $x$ can be is $1^2 + 4^2 + 9^2 = 98 < 100$.  

3
Hence there is no such $x$.

**Case 4:** $x \geq 1000$. Let $x = b_m b_{m-1} \cdots b_0$ where $1 \leq b_m \leq 9$, for $0 \leq i \leq m - 1$, $0 \leq b_i \leq 9$, and $m \geq 3$. Since $b_m \geq 1$ we have $x \geq 10^m$. Since $x$ is the sum of the squares of its digits, $x \leq 81(m+1)$ Summing up

$$10^{m+1} \leq x \leq 81(m+1).$$

We leave it to the reader to show that if $m \geq 3$ then this is impossible.

**Number of Checks:** 2 for Case 2, but 0 for the rest. So just 2.

---

## 4 When is a 2-digit Number the Sum of the Cubes of its Digits

**Theorem 4.1** If $11 \leq x \leq 99$ then $x$ is not the sum of the cubes of its digits.

**Proof:**

Assume, by way of contradiction, that there exists $1 \leq a \leq 9$, $0 \leq b \leq 9$ such that

$$10a + b = a^3 + b^3$$

henceforth the main equation.

Hence $10a + 9 \geq a^3$ so $a^3 - 10a - 9 \leq 0$. We leave it to the reader to show that this implies $a \leq 3$.

Take the main equation mod 2, using $a^3 \equiv a \pmod 2$, $b^3 \equiv b \pmod 2$, $10 \equiv 0 \pmod 2$ to obtain

$$a \equiv 0 \pmod 2.$$ 

Since $1 \leq a \leq 3$ and $a \equiv 0 \pmod 2$, we have $a = 2$. Plugging $a = 2$ into the main equation we get

$$b(b - 1)(b + 1) = 12$$

Alas, 12 cannot be written as the product of three consecutive numbers. Contradiction.

**Number of Checks:** 0.

---

## 5 When is a 3-digit Number the Sum of the Cubes of its Digits

**Theorem 5.1** There is exactly one number $100 \leq x \leq 999$ that is the sum of the cubes of its digits. That number is 153.

**Proof:**

Assume that there exists $1 \leq a \leq 9$, $0 \leq b \leq 9$, $0 \leq c \leq 9$ such that

$$100a + 10b + c = a^3 + b^3 + c^3$$

henceforth the main equation.

We will obtain conditions on $a, b, c$ that force $a = 1, b = 5, c = 3$. 

---
Claim 1: \( a \equiv b \pmod{2} \).

**Proof of Claim 1**

Take the main equation \( \pmod{2} \) to get \( a + b + c \equiv c \pmod{2} \). Hence \( a \equiv b \pmod{2} \).

**End of Proof of Claim 1**

Claim 2:

1. \( b, c \leq 7 \)
2. If \( b \leq 4 \) then \( c \geq 5 \).
3. If \( c \geq 7 \) then \( b \leq 4 \).
4. Assume \( 3 \leq a \leq 8 \). If \( b \leq 6 \) then \( c \geq 5 \).

**Proof of Claim 2**

Note that \( b^3 + c^3 - 10b - c = 100a - a^3 \).

We will need the following table.

<table>
<thead>
<tr>
<th>( a )</th>
<th>( 100a - a^3 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>99</td>
</tr>
<tr>
<td>2</td>
<td>192</td>
</tr>
<tr>
<td>3</td>
<td>273</td>
</tr>
<tr>
<td>4</td>
<td>336</td>
</tr>
<tr>
<td>5</td>
<td>375</td>
</tr>
<tr>
<td>6</td>
<td>384</td>
</tr>
<tr>
<td>7</td>
<td>357</td>
</tr>
<tr>
<td>8</td>
<td>288</td>
</tr>
<tr>
<td>9</td>
<td>171</td>
</tr>
</tbody>
</table>

By the table if \( 1 \leq a \leq 9 \) then \( 99 \leq b^3 + c^3 - 10b - c \leq 384 \).

By the table if \( 3 \leq a \leq 8 \) then \( 273 \leq b^3 + c^3 - 10b - c \leq 384 \).

1) If \( b \geq 8 \) or \( c \geq 8 \) then clearly \( b^3 + c^3 - 10b - c \geq 385 \). Hence \( b, c \leq 7 \).
2) If \( b \leq 4 \) then \( b^3 - 10b \leq 24 \) hence \( 99 \leq c^3 - c + 24 \). Therefore \( c^3 - c \geq 75 \). Hence \( c \geq 5 \).
3) If \( c \geq 7 \) and \( b \geq 5 \) then \( b^3 + c^3 - 10b - c \geq 342 + 75 \geq 384 \). Hence if \( c \geq 7 \) then \( b \leq 4 \).
4) (\( 3 \leq a \leq 8 \)) If \( b \leq 6 \) then \( b^3 - 10b \leq 156 \) hence \( 273 \leq c^3 - c + 156 \). Therefore \( c^3 - c \geq 117 \). Hence \( c \geq 5 \).

**End of Proof of Claim 2**

Claim 3:

1. If \( a \equiv b \pmod{4} \) then \( c \notin \{2, 6\} \).
2. If \( a \not\equiv b \pmod{4} \) and \( a \) is odd then \( c \in \{2, 6\} \).

**Proof of Claim 3:**

1) We take the main equation \( \pmod{4} \) using that \( a^3 \equiv b^3 \pmod{4} \).

\[
2b + c \equiv 2b^3 + c^3 \pmod{4}.
\]
\[ c^3 - c \equiv 2b - 2b^3 \equiv 2b(1 - b)(1 + b) \pmod{4}. \]

Since one of \( b, 1 - b, 1 + b \) is even we have

\[ c^3 - c \equiv 0 \pmod{4}. \]

This is not satisfied by any \( c \in \{2, 6\} \).

1) We take the main equation mod 4 using that \( a^3 + b^3 \equiv 0 \pmod{4} \).

\[ 2b + c \equiv c^3 \pmod{4}. \]

\[ 2b \equiv c^3 - c \pmod{4}. \]

Since \( a \) is odd, \( b \) is odd. The set of \( c \) that satisfy this is \( \{2, 6\} \).

End of Proof of Claim 3

Claim 4: If \( a \equiv 0 \pmod{2} \) then there is no solution to the main equation.

Proof of Claim 4:

If \( a \equiv 0 \pmod{2} \) then, by Claim 1, \( b \equiv 0 \pmod{2} \).

The main equation mod 8, using that \( a^3 \equiv b^3 \equiv 100 \equiv 0 \pmod{8} \) and \( 10 \equiv 2 \pmod{8} \), is

\[ 2b \equiv c^3 - c \pmod{8}. \]

We now present a table of \( c^3 - c \pmod{8} \) as \( 0 \leq c \leq 7 \), and the possible values of \( b \).

<table>
<thead>
<tr>
<th>( c )</th>
<th>( c^3 - c \pmod{8} )</th>
<th>( b )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>0, 4</td>
</tr>
<tr>
<td>1</td>
<td>0</td>
<td>0, 4</td>
</tr>
<tr>
<td>2</td>
<td>6</td>
<td>3</td>
</tr>
<tr>
<td>3</td>
<td>0</td>
<td>0, 4</td>
</tr>
<tr>
<td>4</td>
<td>4</td>
<td>2</td>
</tr>
<tr>
<td>5</td>
<td>0</td>
<td>2</td>
</tr>
<tr>
<td>6</td>
<td>2</td>
<td>1</td>
</tr>
<tr>
<td>7</td>
<td>0</td>
<td>0, 4</td>
</tr>
</tbody>
</table>

For \( 0 \leq c \leq 6 \) we need, by Claim 2, \( b \geq 6 \). Hence none of these values for \( c \) work. The only case left is \( c = 7 \).

If \( c = 7 \) then

\[ 100a + 10b + 7 = a^3 + b^3 + 343. \]

\[ 100a + 10b = a^3 + b^3 + 336. \]

- Since \( 100a + 10b \geq 336 \) we have \( a \geq 3 \). Since \( a \) is even \( a \geq 4 \).
For this item all mods are mod 10. If we take the equation

\[ 100a + 10b = a^3 + b^3 + 336 \]

mod 10 we get

\[ a^3 + b^3 \equiv 4. \]

<table>
<thead>
<tr>
<th>( a )</th>
<th>( a^3 \pmod{10} )</th>
<th>( b^3 = 4 - a^3 \pmod{10} )</th>
<th>( b )</th>
</tr>
</thead>
<tbody>
<tr>
<td>4</td>
<td>4</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>6</td>
<td>6</td>
<td>8</td>
<td>2</td>
</tr>
<tr>
<td>8</td>
<td>2</td>
<td>8</td>
<td>8</td>
</tr>
</tbody>
</table>

The \( a = 8 \) case has \( c = 7 \) and \( b = 8 \) which cannot happen by Claim 3. Hence the only candidates to check are \((4,0,7)\), \((6,0,7)\). None of them work.

**Note:** This claim took 2 checks.

**End of Proof of Claim 4**

We can now enumerate all of the possibilities left. By Claim 4 we need only look at \( a \equiv 1 \pmod{2} \). By Claims 1 and 2 we can assume \( b \in \{1, 3, 5, 7\} \) and \( c \leq 7 \).

1. \( a = 1 \): By Claim 2 if \( b \leq 4 \) then \( c \geq 5 \). Since \( 6^3, 7^3 \geq 200 \) we have \( c \leq 5 \). Since \( 5^3 + 5^3 \geq 200 \) we have that \((a,b,c) \neq (1,5,5)\). Hence the possible \((a,b,c)\) are

   \((1,1,5), (1,3,5), (1,5,0), (1,5,1), (1,5,2), (1,5,3), (1,5,4)\).

   By Claim 3 \((1,3,5)\) and \((1,5,2)\) is eliminated.

   Hence the only ones left to check are

   \((1,1,5), (1,5,0), (1,5,1), (1,5,3), (1,5,4)\).

   Only \((1,5,3)\) works.

   **Note:** This took 5 checks.

2. \( a = 3 \): By Claim 2 if \( b \leq 5 \) then \( c \geq 6 \). Since \( a^3 + 3^3 + 6^3 \leq 299 \) \((3,b,c) \notin \{(3,1,6), (3,3,6)\}\).

   Since \( a^3 + 6^3 + 6^3 \geq 400 \) \((a,b,c) \notin (3,6,6)\).

   Since \( a^3 + 4^3 + 7^3 \geq 499 \)

   \((a,b,c) \notin \{(3,5,7), (3,7,4), (3,7,5), (3,7,6), (3,7,7)\}\).

   Hence the possible \((a,b,c)\) are

   \{(3,1,7), (3,3,7), (3,5,6), (3,7,0), (3,7,1), (3,7,2), (3,7,3)\}.

   By Claim 3 \((3,1,7)\) and \((3,7,2)\) are eliminated.

   Hence the only ones left to check are
None of these work.

This took 5 checks.

3. $a = 5$. By Claim 2 if $b \leq 5$ then $c \geq 6$. Since $a^3 + 5^3 + 6^3 \leq 499$ $(a, b, c) \notin \{(5, 1, 6), (5, 3, 6), (5, 5, 6)\}$. Since $a^3 + 3^3 + 7^3 \geq 499$

$$(a, b, c) \notin \{(5, 7, 3), (5, 7, 4), (5, 7, 5), (5, 7, 6), (5, 7, 7), (5, 3, 7), (5, 5, 7)\}.$$

The only cases left to check is $(a, b, c) = \{(5, 1, 7), (5, 7, 1)\}$. We can eliminate $(5, 7, 1)$ by Claim 3. Hence the only case to check is $(5, 1, 7)$. This case does not work.

Note: This took 1 check.

4. $a = 7$. By Claim 2 if $b \leq 5$ then $c \geq 6$. Since $a^3 + 1^3 + 7^3 \leq 699$, $(a, b, c) \notin \{(7, 1, 6), (7, 1, 7)\}$. Since $a^3 + 5^3 + 6^3 \leq 699$, $(a, b, c) \notin \{(7, 3, 6), (7, 5, 6)\}$. Since $a^3 + 5^3 + 7^3 \geq 801$, $(a, b, c) \notin \{(7, 5, 7), (7, 7, 6), (7, 7, 7)\}$. The only cases left to check are $(a, b, c) \in \{(7, 3, 7), (7, 7, 7)\}$.

By Claim 3 $(7, 3, 7)$ and $(7, 7, 7)$ are eliminated.

Note: This took 0 checks.

5. $a = 9$. By Claim 2 if $b \leq 4$ then $c \geq 5$. Since $a^3 + 7^3 \geq 1000$ we have $b \leq 5$ and $c \leq 6$. Since $a^3 + 5^3 + 3^3 \leq 899$ we have $(a, b, c) \notin \{(9, 5, 0), (9, 5, 1), (9, 5, 2), (9, 5, 3)\}$. The only cases left to check are $(a, b, c) \in \{(9, 1, 5), (9, 1, 6), (9, 3, 5), (9, 3, 6), (9, 5, 4), (9, 5, 5), (9, 5, 6)\}$.

By Claim 3 $(9, 1, 6), (9, 3, 5), (9, 5, 6)$ are eliminated.

Hence the only cases left to check are

$$(a, b, c) \in \{(9, 1, 5), (9, 3, 6), (9, 5, 4), (9, 5, 5)\}.$$

None of these work.

Note: this took 4 checks.

Note: The total number of checks is 24.

Question: Is there a proof of Theorem 5.1 with less checks?