When Does a Number Equal the Sum of the Squares or Cubes of its Digits?
An Exposition and a Call for a More elegant Proof

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1 Introduction

We will look at theorems of the following form:

Theorem 1.1 *Let $L < U$ and k be in N. For every $L \le x \le U$ there are XXX numbers that are the sum of the kth powers of their digits.*

The theorems we prove are easy and known. We will give several proofs of some of them with an eye towards the question: What is an elegant proof? For the last theorem I have a proof that is not elegant. If you know a more elegant one then please let me know.

For each proof we note how many times we had to actually CHECK a number. This is a crude measure of elegance— the fewer checks the more elegant the proof (compared to other proofs of the same theorem). We try not to *game the system* by having rather elaborate proofs that don't check that much.

2 When is a 2-Digit Number the Sum of the Squares of its Digits?

We will prove the following theorem several different ways.

Theorem 2.1 *If $11 \le x \le 99$ then x is not the sum of the squares of its digits.*

2.1 Proof by Enumeration

One can prove this theorem by listing all numbers between 11 and 99 and seeing that none of them work. One could write a program to produce this proof. We omit this proof.
Number of Checks: : 89.

PROS: You get the answer easily. If you wrote a program then you can adjust it to find the answers. to many more problems of this type (e.g., How many 3-digits numbers are the sum of the cubes of their digits) or variants of this problem (e.g., How many 2-digit numbers are within 1 of the sum of their digits? Which ones are they?)
CONS: You get no insights.

2.2 Proof by Looking at each 10-number interval

Proof:
Case 1: $x = 10a$ where $1 \le a \le 9$ (so x ends in a 0).

$$a^2 = 10a$$

$$a = 10.$$

So this cannot happen. Note we did no checks.
Case 2: $11 \le x \le 19$. Since $11, 12, 13 > 1^2 + 3^2 > 1^2 + 2^2 > 1^2 + 1^1$, none of 11,12,13 work. Since $15, 16, 17, 18, 19 < 1^2 + 5^2 < 1^2 + 6^2 < 1^2 + 7^2 < 1^2 + 8^2 < 1^2 + 9^2$, none of 15,16,17,18,19 work. The only number left is 14 which does not work since $14 \neq 1^2 + 4^2$. Note we did one check.

Case 3: Look at $21 \le x \le 29$. $21, 22, 23, 24 > 2^2 + 4^2$. $26, 27, 28, 29 < 2^2 + 6^2$. 25 does not work since $25 \neq 2^2 + 5^2$. Note that we did one check.

Case 4: $31 \le x \le 39$. $31, 32, 33, 34 > 3^2 + 4^2$. $36, 37, 38, 39 < 3^2 + 6^2$. $35 \neq 3^2 + 5^2$. Note that we did zero checks.

Case 5: $41 \le x \le 49$. $41, 42, 43, 44 > 4^2 + 4^2$. $46, 47, 48, 49 < 4^2 + 6^2$. $45 \neq 4^2 + 5^2$. Note that we did one check.

Case 6: $51 \le x \le 59$. $51, 52, 53, 54, 55 > 5^2 + 5^2$. $56, 57, 58, 59 < 5^2 + 9^2$. Note that we did zero checks.

Case 7: $61 \le x \le 69$. $61, 62, 63, 64 > 6^2 + 4^2$. $66, 67, 68, 69 < 6^2 + 6^2$. $65 \neq 6^2 + 5^2$. Note that we did one check.

Case 8: $71 \le x \le 79$. $71, 72, 73, 74 > 7^2 + 4^2$. $76, 77, 78, 78 < 7^2 + 6^2$. $75 \neq 7^2 + 5^2$. Note that we did one check.

Case 9: $81 \le x \le 89$. $81, 82, 83, 84 > 8^2 + 4^2$. $86, 87, 88, 89 < 8^2 + 6^2$. $85 \neq 8^2 + 5^2$. Note that we did one check.

Case 9: $91 \le x \le 99$. $91, 92 > 9^2 + 2^2$ $95, 96, 97, 98, 99 < 9^2 + 5^2$. $93 \neq 9^2 + 3^2$, $94 \neq 9^2 + 4^2$. Note that we did two checks.

Number of Checks: 8

PRO: Only ten cases. The techniques used may be helpful for other problems.
CON: Only ten cases? If you do other problems with way the number of cases may get rather large.
$\blacksquare$

2.3 Proof Using Mod 2, Mod 4, and Mod 10

Proof:
Assume, by way of contradiction, that there exists $1 \le a \le 9$ and $0 \le b \le 9$ such that

$$10a + b = a^2 + b^2 \text{ henceforth } \textit{the main equation.}$$

Take this mod 2 using $x^2 \equiv x \pmod 2$ and $10 \equiv 0 \pmod 2$ to obtain

$$a \equiv 0 \pmod 2.$$

If we take the main equation mod 4, using $a^2 \equiv 0 \pmod{4}$, we get

$$b(b-1) \equiv 0 \pmod{4}.$$

Hence $b \in \{0, 1, 4, 5, 8, 9\}$.
We now take the main equation mod 10.

$$b \equiv a^2 + b^2 \pmod{10}.$$

$$a = \sqrt{-b(b-1)} \pmod{10}.$$

We now run through $b \in \{0, 1, 4, 5, 8, 9\}$ and note for which ones $a = \sqrt{-b(b-1)} \pmod{10}$ exists and is a nonzero even number. Note that the squares mod 10 are $0, 1, 4, 5, 6, 9$ so the only nonzero even ones are $4, 6$.

b	$-b(b-1) \pmod{10}$	$a = \sqrt{-b(b-1)} \pmod{10}$	comment
0	0	0	NO GOOD: Need $1 \le 1 \le 9$
1	0	0	NO GOOD: Need $1 \le 1 \le 9$
4	8	NONE	NO GOOD: No Square Root
5	0	0	NO GOOD: Need $1 \le 1 \le 9$
6	0	0	NO GOOD: Need $1 \le 1 \le 9$
8	4	$2, 8$	OH, Will need to check
9	8	NONE	NO GOOD: No Square Root

So the only candidates for (a, b) are $(2, 4), (8, 4)$.
One can check that these do not satisfy the main equation.
Number of Checks: 2
PRO: The technique may extends to other problems.
CAVEAT: Is this proof really more elegant then the intervals proof?
∎

3 When is a d-Digit Number the Sum of the Squares of its Digits?

Theorem 3.1 *If $x \ge 2$ then x is not the sum of the squares of its digits.*

Proof:
Case 1: $2 \le x \le 9$. Since $x < x^2$ the theorem is true for $2 \le x \le 9$.
Case 2: $10 \le x \le 99$. By Theorem 2.1 our theorem is true for these x.
Case 3: $1 \le a \le 9$, $0 \le b \le 9$, $0 \le c \le 9$. Assume

$$x = 100a + 10b + c = a^2 + b^2 + c^2.$$

The largest x can be is $9^2 + 9^2 + 9^2 < 300$. Hence $a \in \{1, 2\}$.
The largest x can be is $2^2 + 9^2 + 9^2 = 176 < 200$. Hence $a = 1$ and $b \le 7$.
The largest x can be is $1^2 + 7^2 + 9^2 = 141$. Hence and $b \le 4$.
The largest x can be is $1^2 + 4^2 + 9^2 = 98 < 100$.

Hence there is no such x.

Case 4: $x \geq 1000$. Let $x = b_m b_{m-1} \cdots b_0$ where $1 \leq b_m \leq 9$, for $0 \leq i \leq m-1$, $0 \leq b_i \leq 9$, and $m \geq 3$. Since $b_m \geq 1$ we have $x \geq 10^m$. Since x is the sum of the squares of its digits, $x \leq 81(m+1)$ Summing up

$$10^{m+1} \leq x \leq 81(m+1).$$

We leave it to the reader to show that if $m \geq 3$ then this is impossible.

Number of Checks: 2 for Case 2, but 0 for the rest. So just 2.

▮

4 When is a 2-digit Number the Sum of the Cubes of its Digits

Theorem 4.1 *If* $11 \leq x \leq 99$ *then* x *is not the sum of the cubes of its digits.*

Proof:

Assume, by way of contradiction, that there exists $1 \leq a \leq 9$, $0 \leq b \leq 9$ such that

$$10a + b = a^3 + b^3 \text{ henceforth } \textit{the main equation.}$$

Hence $10a + 9 \geq a^3$ so $a^3 - 10a - 9 \leq 0$. We leave it to the reader to show that this implies $a \leq 3$.

Take the main equation mod 2, using $a^3 \equiv a \pmod 2$, $b^3 \equiv b \pmod 2$, $10 \equiv 0 \pmod 2$ to obtain

$$a \equiv 0 \pmod 2.$$

Since $1 \leq a \leq 3$ and $a \equiv 0 \pmod 2$, we have $a = 2$. Plugging $a = 2$ into the main equation we get

$$b(b-1)(b+1) = 12$$

Alas, 12 cannot be written as the product of three consecutive numbers. Contradiction.

Number of Checks: 0.

▮

5 When is a 3-digit Number the Sum of the Cubes of its Digits

Theorem 5.1 *There is exactly one number* $100 \leq x \leq 999$ *that is the sum of the cubes of its digits. That number is 153.*

Proof:

Assume that there exists $1 \leq a \leq 9$, $0 \leq b \leq 9$, $0 \leq c \leq 9$ such that

$$100a + 10b + c = a^3 + b^3 + c^3 \text{ henceforth } \textit{the main equation.}$$

We will obtain conditions on a, b, c that force $a = 1, b = 5, c = 3$.

Claim 1: $a \equiv b \pmod 2$.

Proof of Claim 1

Take the main equation mod 2 to get $a + b + c \equiv c \pmod 2$. Hence $a \equiv b \pmod 2$.

End of Proof of Claim 1

Claim 2:

1. $b, c \leq 7$
2. If $b \leq 4$ then $c \geq 5$.
3. If $c \geq 7$ then $b \leq 4$.
4. Assume $3 \leq a \leq 8$. If $b \leq 6$ then $c \geq 5$.

Proof of Claim 2

Note that $b^3 + c^3 - 10b - c = 100a - a^3$.

We will need the following table.

a	$100a - a^3$
1	99
2	192
3	273
4	336
5	375
6	384
7	357
8	288
9	171

By the table if $1 \leq a \leq 9$ then $99 \leq b^3 + c^3 - 10b - c \leq 384$.

By the table if $3 \leq a \leq 8$ then $273 \leq b^3 + c^3 - 10b - c \leq 384$.

1) If $b \geq 8$ or $c \geq 8$ then clearly $b^3 + c^3 - 10b - c \geq 385$. Hence $b, c \leq 7$.

2) If $b \leq 4$ then $b^3 - 10b \leq 24$ hence $99 \leq c^3 - c + 24$. Therefore $c^3 - c \geq 75$. Hence $c \geq 5$.

3) If $c \geq 7$ and $b \geq 5$ then $b^3 + c^3 - 10b - c \geq 342 + 75 > 384$. Hence if $c \geq 7$ then $b \leq 4$.

4) $(3 \leq a \leq 8)$ If $b \leq 6$ then $b^3 - 10b \leq 156$ hence $273 \leq c^3 - c + 156$. Therefore $c^3 - c \geq 117$. Hence $c \geq 5$.

End of Proof of Claim 2

Claim 3:

1. If $a \equiv b \pmod 4$ then $c \notin \{2, 6\}$.
2. If $a \not\equiv b \pmod 4$ and a is odd then $c \in \{2, 6\}$.

Proof of Claim 3:

1) We take the main equation mod 4 using that $a^3 \equiv b^3 \pmod 4$.

$$2b + c \equiv 2b^3 + c^3 \pmod 4.$$

$$c^3 - c \equiv 2b - 2b^3 \equiv 2b(1-b)(1+b) \pmod 4.$$

Since one of $b, 1-b, 1+b$ is even we have

$$c^3 - c \equiv 0 \pmod 4.$$

This is not satisfied by any $c \in \{2, 6\}$.

1) We take the main equation mod 4 using that $a^3 + b^3 \equiv 0 \pmod 4$.

$$2b + c \equiv c^3 \pmod 4.$$

$$2b \equiv c^3 - c \pmod 4.$$

Since a is odd, b is odd. The set of c that satisfy this is $\{2, 6\}$.

End of Proof of Claim 3

Claim 4: If $a \equiv 0 \pmod 2$ then there is no solution to the main equation.

Proof of Claim 4:

If $a \equiv 0 \pmod 2$ then, by Claim 1, $b \equiv 0 \pmod 2$.

The main equation mod 8, using that $a^3 \equiv b^3 \equiv 100 \equiv 0 \pmod 8$ and $10 \equiv 2 \pmod 8$, is

$$2b \equiv c^3 - c \pmod 8.$$

We now present a table of $c^3 - c \pmod 8$ as $0 \le c \le 7$, and the possible values of b.

c	$c^3 - c \pmod 8$	b
0	0	0, 4
1	0	0, 4
2	6	3
3	0	0, 4
4	4	2
5	0	2
6	2	1
7	0	0, 4

For $0 \le c \le 6$ we need, by Claim 2, $b \ge 6$. Hence none of these values for c work. The only case left is $c = 7$.

If $c = 7$ then

$$100a + 10b + 7 = a^3 + b^3 + 343.$$

$$100a + 10b = a^3 + b^3 + 336.$$

- Since $100a + 10b \ge 336$ we have $a \ge 3$. Since a is even $a \ge 4$.

- For this item all mods are mod 10. If we take the equation

$$100a + 10b = a^3 + b^3 + 336$$

mod 10 we get

$$a^3 + b^3 \equiv 4.$$

a	$a^3 \pmod{10}$	$b^3 = 4 - a^3 \pmod{10}$	b
4	4	0	0
6	6	8	2
8	2	2	8

The $a = 8$ case has $c = 7$ and $b = 8$ which cannot happen by Claim 3. Hence the only candidates to check are $(4, 0, 7)$, $(6, 0, 7)$. None of them work.

Note: This claim took 2 checks.
End of Proof of Claim 4

We can now enumerate all of the possibilities left. By Claim 4 we need only look at $a \equiv 1 \pmod 2$. By Claims 1 and 2 we can assume $b \in \{1, 3, 5, 7\}$ and $c \leq 7$.

1. $a = 1$: By Claim 2 if $b \leq 4$ then $c \geq 5$. Since $6^3, 7^3 \geq 200$ we have $c \leq 5$. Since $5^3 + 5^3 \geq 200$ we have that $(a, b, c) \neq (1, 5, 5)$. Hence the possible (a, b, c) are

$$(1, 1, 5), (1, 3, 5), (1, 5, 0), (1, 5, 1), (1, 5, 2), (1, 5, 3), (1, 5, 4).$$

By Claim 3 $(1, 3, 5)$ and $(1, 5, 2)$ is eliminated.

Hence the only ones left to check are

$$(1, 1, 5), (1, 5, 0), (1, 5, 1), (1, 5, 3), (1, 5, 4).$$

Only $(1, 5, 3)$ works.

Note: This took 5 checks.

2. $a = 3$. By Claim 2 if $b \leq 5$ then $c \geq 6$. Since $a^3 + 3^3 + 6^3 \leq 299$ $(a, b, c) \notin \{(3, 1, 6), (3, 3, 6)\}$. Since $a^3 + 6^3 + 6^3 \geq 400$ $(a, b, c) \neq (3, 6, 6)$. Since $a^3 + 4^3 + 7^3 \geq 499$

$(a, b, c) \notin \{(3, 5, 7), (3, 7, 4), (3, 7, 5), (3, 7, 6), (3, 7, 7)\}$.

Hence the possible (a, b, c) are

$$\{(3, 1, 7), (3, 3, 7), (3, 5, 6), (3, 7, 0), (3, 7, 1), (3, 7, 2), (3, 7, 3)\}.$$

By Claim 3 $(3, 1, 7)$ and $(3, 7, 2)$ are eliminated.

Hence the only ones left to check are

$$\{(3,3,7),(3,5,6),(3,7,0),(3,7,1),(3,7,3)\}.$$

None of these work.

This took 5 checks.

3. $a = 5$. By Claim 2 if $b \le 5$ then $c \ge 6$. Since $a^3+5^3+6^3 \le 499$ $(a,b,c) \notin \{(5,1,6),(5,3,6),(5,5,6)\}$. Since $a^3 + 3^3 + 7^3 \ge 499$

   $$(a,b,c) \notin \{(5,7,3),(5,7,4),(5,7,5),(5,7,7),(5,7,6),(5,7,7),(5,3,7),(5,5,7)\}.$$

   The only cases left to check is $(a,b,c) = \{(5,1,7),(5,7,1)\}$. We can eliminate $(5,7,1)$ by Claim 3. Hence the only case to check is $(5,1,7)$. This case does not work.

   Note: This took 1 check.

4. $a = 7$. By Claim 2 if $b \le 5$ then $c \ge 6$. Since $a^3 + 1^3 + 7^3 \le 699$, $(a,b,c) \notin \{(7,1,6),(7,1,7)\}$. Since $a^3 + 5^3 + 6^3 \le 699$, $(a,b,c) \notin \{(7,3,6),(7,5,6)\}$. Since $a^3 + 5^3 + 7^3 \ge 801$, $(a,b,c) \notin \{(7,5,7),(7,7,6),(7,7,7)\}$. The only cases left to check are $(a,b,c) \in \{(7,3,7),(7,7,7)\}$.

   By Claim 3 $(7,3,7)$ and $(7,7,7)$ are eliminated.

   Note: This took 0 checks.

5. $a = 9$. By Claim 2 if $b \le 4$ then $c \ge 5$. Since $a^3 + 7^3 \ge 1000$ we have $b \le 5$ and $c \le 6$. Since $a^3 + 5^3 + 3^3 \le 899$ we have $(a,b,c) \notin \{(9,5,0),(9,5,1),(9,5,2),(9,5,3)\}$. The only cases left to check are $(a,b,c) \in \{(9,1,5),(9,1,6),(9,3,5),(9,3,6),(9,5,4),(9,5,5),(9,5,6)\}$.

   By Claim 3 $(9,1,6)$, $(9,3,5)$, $(9,5,6)$ are eliminated.

   Hence the only cases left to check are

   $(a,b,c) \in \{(9,1,5),(9,3,6),(9,5,4),(9,5,5)\}$.

   None of these work.

   Note: this took 4 checks.

Note: The total number of checks is 24.
∎

Question: Is there a proof of Theorem 5.1 with less checks?