Envy-Free Discrete Protocols
Exposition by William Gasarch

1 Introduction

Whenever we say something like *Alice has a piece worth* α we mean it's worth α TO HER. The term *biggest piece* means most valuable to the person looking at it. This is not necessarily related to geometric size. We assume the entire cake is worth 1 to everyone.

Def 1.1 An n-person protocol for division is *envy-free* if each person thinks they received the biggest (or tied) piece.

In this document we show

1. There is an envy-free protocol for 3 people that uses at most 5 cuts.
2. For every ϵ there is a protocol for 4 people that will (1) leave $\le \epsilon$ (in everyones view) unallocated, and (2) is envy-free on all that is allocated.
3. There is an envy-free protocol for 4 people. (This protocol easily generalizes to n people.)

Our exposition is based on the paper that first proved there is a discrete envy-free protocol for $n \ge 4$, namely *An Envy-free cake division protocol* by Brams and Taylor [1].

Convention 1.2 Protocols are presented by giving the instructions for what each player *must* do and, in parenthesis, advice for what the player *should do in his or her best interest.* This means that, if there are n people, and (say) Alice doesn't follow the advice then Alice might get $< 1/n$.

2 A 3-Person Discrete Envy-Free Protocol

The following definition will be the KEY to both this protocol and the 4-person envy-free protocol which we present later.

Def 2.1 Assume Alice, Bob, and Carol want to cut a cake (what else is new?). Assume that part of the cake has been allocated: Alice gets piece P_1, Bob gets piece P_2, Carol gets piece P_3, and there is some Trim T left over. Alice *has an advantage over Bob* if, no matter how much of T Bob gets, Alice will not envy him. For example, if Alice has 2/5, Bob has 2/5, Carol has 1/10 and the Trim is 1/10 then Alice would not care if Carol got ALL of the trim. Note that Alice would mind if Bob got more trim then Alice gets.

The following Theorem is due to Conway and Selfridge from about 1960. They never published it; however, it appears in [1].

Theorem 2.2 *There is a discrete protocol for 3 people to achieve an envy-free division.*

Proof:

We call the protocol ENVYFREE3. It is in 2 phases.

PHASE ONE of ENVYFREE3

1. Alice cuts the cake into 3 pieces. (All equal.)
2. Bob trims a piece or not. Put the trimming aside. (Trim to create a tie for the top 2 pieces.)
3. There are now 3 pieces and possibly some Trimming. Call the pieces P_1, P_2, P_3 and the trimming T.
4. Carol takes one of P_1, P_2, P_3. (The biggest piece.)
5. Bob takes one of the pieces that are left. However, if the trimmed piece is left he must take it. (The biggest piece available.)
6. Alice takes the remaining piece.

END OF PHASE ONE of ENVYFREE3

We leave it to the reader to prove that if a player does not follow the advice then he or she might get less than 1/3 of $P_1 \cup P_2 \cup P_3$. Henceforth we assume that all players follow the advice.

Claim 1: The Phase One protocol results in an envy-free division of $P_1 \cup P_2 \cup P_3$. (Note that we still need to deal with T).

Proof of Claim 1:

If Bob did not trim a piece then he thinks that 2 pieces are tied for first. If Bob did trim a piece then, since he trimmed it, he thinks that 2 pieces (now) are tied for first.

Carol will get first pick of P_1, P_2, P_3, so she cannot feel envy. Bob will get second pick but he thinks that 2 of the pieces were tied, so he cannot feel envy. If there was a trimmed piece then either Carol or Bob got that trimmed piece. Hence Alice will get one of the untrimmed pieces. Since Alice originally cut the pieces equally, and she gets an untrimmed piece, she cannot feel envy.

End of Proof of Claim 1

If Bob does not trim a piece then Phase one gives us the Envy-Free Division and we are done.

If Bob trimmed a piece then we do Phase 2 where we split the trimming. We assume that Bob got the trimmed piece (the situation where Carol got the trimmed piece is similar).

There is a KEY DIFFERENCE between the situation we have now and what we had in Phase one. Note that Alice thinks that Bob has a piece with some stuff missing. *If Bob gets some or even all of T then Alice cannot envy him.* Using our definition: Alice has an advantage over Bob.

PHASE TWO OF ENVYFREE3:

1. Carol cuts T into 3 pieces (equally).
2. Bob picks a piece. (Biggest piece)
3. Alice picks a piece. (Biggest piece left)
4. Carol picks a piece. (Whatever is left)

END OF PHASE TWO OF ENVYFREE3:

We leave it to the reader to prove that if a player does not follow the advice then he or she might get less than 1/3 of T. Henceforth we assume that all players follow the advice.

Claim 2: The Phase Two protocol results in an envy-free division of T.

Proof of Claim 2:

Bob cannot feel envy since he got the first choice of pieces.

Alice is the interesting case. Alice cannot feel envy towards Bob since WHATEVER Bob gets, Alice just thinks Bob is making up for having a trimmed piece in the first place. So Alice cannot be envious of Bob. Alice cannot be envious of Carol since Alice got to pick before Carol.

Carol cannot be envious of anyone since she cut T into 3 equal pieces.

End of Proof of Claim 2

Since both Phase one and Phase 2 are envy-free, the entire procedure is envy-free.

▮

Note that the above algorithm took at most 5 cuts. I believe it is an open question as to whether there is a discrete envy-free protocol for 3 people that uses 4 cuts. We know there cannot be a discrete envy-free protocol for 4 people with 3 cuts since there is no proportional protocol with 3 cuts.

Is there a discrete envy-free algorithm for 4 players? 5? n? YES. We will present the 4-person envy-free protocol later in this document. This protocol contains all of the ideas for general n.

3 An all-but-ϵ Envy-Free Protocol for 4 People

The protocol in this section will later be used in the 4 person Envy-Free Protocol. I do not know who first came up with it; however, it appears[1] in [1].

Def 3.1 An *all-but-ϵ envy-free protocol* divides the cake, *except* a piece that everyone agrees is $\leq \epsilon$, in an envy-free way. The piece of size $\leq \epsilon$ is not allocated.

Theorem 3.2 *For every $\epsilon > 0$ there is an all-but-ϵ envy-free protocol for 4 people.*

[1] The protocol is actually used inside a rather large protocol so it is not isolated out.

Proof: We will use FIRST-PERSON-HAPPY (below) as a part of our final protocol. FIRST-PERSON HAPPY will explicitly take as input A, B, C, D meaning Alice, Bob, Carol, Donna, and also P meaning a pie (or a piece of a pie). This is so we can later run it with different people and different pieces later.

FIRST-PERSON-HAPPY(A,B,C,D;P)

1. Alice cuts P into 5 pieces (all equal). YES, its 5 not 4.
2. Bob trims at most 2 of the pieces (to create a 3-way tie for the biggest). The trimming is put aside.
3. Carol trims at most one piece (to create 2-way tie for biggest piece). The trimming is put aside.
4. Donna takes a piece.
5. Carol takes a piece. If the piece she trimmed is available she must take it.
6. Bob takes a piece. If a piece he trimmed is available he must take it.
7. Alice takes a piece.

END OF FIRST-PERSON-HAPPY

This is not a division of the entire cake. There is Trim left over. It is possible that (say) Bob thinks the trim is worth 9/10 of the cake!

We leave it to the reader to show that if someone does not follow the advice they could end up with less than 1/4 of what is divided. Henceforth we assume that everyone follows the advice.

Claim 1:

1. Alice gets 1/5. Hence Alice thinks that there is 4/5 left.
2. Let X be the part of the cake that was shared by all (so this is the cake minus the trim set side). The division of X is envy-free.

Proof:

We look at all the players.

Alice: Alice initially splits the cake into 5 equal pieces. At most 3 pieces were trimmed, hence there are at least 2 untrimmed pieces. Since Bob and Carol must take pieces they trimmed if they are available, Alice will get one of the untrimmed pieces. Hence Alice gets 1/5 and thinks everyone else got $\le 1/5$. Therefore Alice cannot envy anyone.

Bob: Bob created a 3-way tie for first, say the equal pieces are P_1, P_2, P_3. If he gets any of those he will not envy anyone. We consider 2 cases.

- Carol trims a piece, say P_3. So now Bob thinks that P_1 and P_2 are tied for first and that P_3 is smaller. Since Carol must pick P_3 if it available, one of Donna or Carol chooses P_3. The other one may choose one of P_1 or P_2 (or not); however, Bob will be able to get one of P_1 or P_2.

- Carol does not trim P_1 or P_2 (she might trim some piece or NO piece). After Donna and Carol choose, at least one of P_1, P_2, P_3 is available. So Bob will get one of P_1, P_2, P_3.

Carol: Carol created a 2-way tie and she goes second, so she'll be able to get one of the top pieces.

Donna: She goes first. She's happy as a clam!

End of Proof of Claim 1

Note that Alice has 1/5 of P but the others could have really small parts of P. It is possible that Bob thinks he has 1/100 and everyone else has $\le 1/100$.

In protocol ALL-HAPPY they all take turns being Alice.

ALL-HAPPY(A,B,C,D;P)

1. Run FIRST-PERSON-HAPPY$(A, B, C, D; P)$. Let T_1 be the trimming that was set aside.

2. Run FIRST-PERSON-HAPPY$(B, C, D, A; T_1)$. Let T_2 be the trimming that was set aside.

3. Run FIRST-PERSON-HAPPY$(C, D, A, B; T_2)$. Let T_3 be the trimming that was set aside.

4. Run FIRST-PERSON-HAPPY$(D, A, B, C; T_3)$. Set aside the trimmings which we call T.

END OF ALL-HAPPY

Claim 2: If ALL-HAPPY is run then

1. Everyone thinks T has value $\leq \frac{4}{5}$.

2. Let X be the part of the cake that was shared by all (so $X = P - T$). X is divided in an envy-free way.

Proof of Claim 2:

We consider each player in turn. We use the obvious fact that everyone thinks

$$T \leq T_3 \leq T_2 \leq T_1 \leq P.$$

Alice: She thinks $T \leq T_1 \leq \frac{4P}{5}$.

Bob: He thinks $T \leq T_2 \leq \frac{4T_1}{5} \leq \frac{4P}{5}$.

Carol: She thinks $T \leq T_3 \leq \frac{4T_2}{5} \leq \frac{4P}{5}$.

Donna: She thinks $T \leq \frac{4T_3}{5} \leq \frac{4P}{5}$.

The protocol is Envy-Free on the non-trim parts since FIRST-PERSON-HAPPY is Envy-free on the non-trim parts.

End of Proof of Claim 2

We now present the final protocol. It essentially keeps running ALL-HAPPY until T gets to be of size $\leq \epsilon$.

ALMOST-ENVY-FREE4$(A, B, C, D; P; \epsilon)$

1) Let m be such that $(\frac{4}{5})^m < \epsilon$.

2) Run ALL-HAPPY$(A, B, C, D; P)$. Let T_1 be the trimming that was set aside. Note that everyone thinks $T_1 \leq \frac{4}{5}$.

3) Run ALL-HAPPY(A, B, C, D, T_1). Let T_2 be the trimming that was set aside. Note that everyone thinks $T_2 \leq (\frac{4}{5})^2$.

4) Run ALL-HAPPY(A, B, C, D, T_2). Let T_3 be the trimming that was set aside. Note that everyone thinks $T_3 \le (\frac{4}{5})^3$. $\vdots$

m+1) Run ALL-HAPPY(A, B, C, D, T_{m-1}). Let T_m be the trimming that was set aside. Note that everyone thinks $T_m \le (\frac{4}{5})^m < \epsilon$. Let $T = T_m$ be the trimming of that was set aside. Note that everyone thinks $T < \epsilon$.

END OF ALMOST-ENVY-FREE4

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4 A 4-Person Discrete Envy-Free Protocol: First Attempt

We will attempt a 4-person Discrete Envy-Free Protocol just to see what subprotocols we will need. We will revisit this sketch of a protocol a few times before giving the final full protocol.

Theorem 4.1 *There is a discrete protocol for 4 people that achieves an envy-free division.*

Attempt at Protocol

The players are Alice, Bob, Carol, Donna.

1. Alice cuts the cake into 4 pieces P, Q, R, S. (All equal.)
2. Bob, Carol, and Donna each write down on a separate piece of paper E (if they think the pieces are all equal) or N (if they think they are not all equal). We assume that Alice thinks they are equal.
3. If Alice, Bob, Carol, Donna all write E then we give P to Alice, Q to Bob, R to Carol, and R to Donna, and we are done. Note that any distribution would have worked.
4. If not then what do we know? We know that at least one person thinks the cake was split unevenly. Say Bob thinks $P > Q$. Alice thinks $P = Q$. Can we use this?

If 2 players disagree then we can use that. In the next 2 sections we give protocols for just 2 people. These will be used in our final protocol and hence do not allocate any of the cake.

4.1 Making Alice and Bob REALLY Disagree

Lemma 4.2 *Assume that Alice and Bob are both looking at pieces P, Q and Alice things $P = Q$ while Bob thinks $P > Q$. There is a protocol that produces P', Q' such that*

- *Alice thinks $P' < Q'$.*
- *Bob thinks $P' > Q'$.*
- *$P \cup Q = P' \cup Q'$.*

Proof:
MAKE-UNEQUAL

1. Bob cuts the cake into a number m of pieces where Bob picks m. (m is picked so large that if Bob cuts the piece into m pieces and Alice takes any piece, Bob still thinks he has more. Bob cuts the cake into m equal pieces.)
2. Alice takes one of the pieces. (Alice takes the largest piece.)

END OF MAKE-UNEQUAL ▮

Exercise 1 Let p be how much Bob values P. Let q be how much Bob values Q. Determine a good value of m as a function of p, q.

4.2 Making Alice and Bob Have an Advantage Over Each Other

Lemma 4.3 *Assume that Alice and Bob are both looking at pieces P, Q and Alice thinks $P = Q$ while Bob thinks $P > Q$. There is a protocol that produces (likely very small) pieces $p_1, p_2, p_3, q_1, q_2, q_3$ such that*

- *Alice thinks $q_1 = q_2 = q_3 > p_1, p_2, p_3$.*
- *Bob thinks $p_1 = p_2 = p_3 > q_1, q_2, q_3$.*

Proof:

In the protocol SIX FUNKY PIECES we often let P (or Q, P_i, Q_i stand for either the piece, what Bob thinks it is worth, and what Alice thinks it is worth. Which one is intended will be clear from context.

SIX FUNKY PIECES

1. Alice and Bob run the protocol from Lemma 4.2 to obtain P', Q' with $P' < Q'$ and $P' > Q'$. For notation we rename them and assume Alice thinks $P < Q$ and Bob thinks $P > Q$.

2. Bob names a number $m \geq 10$ (m should be big enough so that no matter how P is cut into m pieces, if Bob discards the SIX smallest pieces, then he still thinks he has more than Alice.

3. Alice cuts P into m pieces $P_1, \ldots, P_m$ and Q into m pieces $Q_1, \ldots, Q_m$. (Cuts them both equally. Note that Alice will think

$$P_1 = \cdots = P_m < Q_1 = \cdots = Q_m.$$

)

4. Bob sorts the pieces:

   (a) Bob thinks that $P_m \leq P_{m-1} \leq \cdots \leq P_3 \leq P_2 \leq P_1$.

   (b) Bob thinks that $Q_m \leq Q_{m-1} \leq \cdots \leq Q_3 \leq Q_2 \leq Q_1$.

5. If Bob thinks $P_3 > Q_{m-2}$ then

   (a) Bob trims P_1, P_2 (down to P_3 value). Let the trimmed versions be P_1', P_2'. Let $p_1 = P_1'$, $p_2 = P_2'$, $p_3 = P_3$.

   (b) Let $q_1 = Q_{m-2}$, $q_2 = Q_{m-1}$, $q_3 = Q_m$.

   Why this works:

   - Bob thinks $p_1 = p_2 = p_3$ and, since he thinks $P_3 > Q_{m-2}$ he thinks

   $$p_1 = p_2 = p_3 = P_3 > \{Q_{m-2} \geq Q_{m-1} \geq Q_m\} = \{q_1 \leq q_2 \leq q_3\}.$$

   - Alice thinks

   $$Q_m = q_1 = q_2 = q_3 > P_1 \geq P_1' = p_1 = p_2 = p_3.$$

6. If Bob thinks $P_3 \le Q_{m-2}$ then what do we do? INTUITION: Since Bob thinks

   (1) $P_3 \le Q_{m-2}$,

   (2) the union of all of the P_i's is more than the union of all of the Q_i's,

   (3) P_3 is the third largest P_i,

   Bob must think P_1 is really big!.

   (a) Bob cuts P_1 into 3 (equal) pieces. Call them p_1, p_2, p_3.
   (b) Let $q_1 = Q_m$, $q_2 = Q_{m-1}$, $q_2 = Q_{m-2}$.

   We show later why this works.

END OF SIX FUNKY PIECES

The only case we didn't prove works was the last one. For Alice this is easy. She thinks

$$q_1 = q_2 = q_3 > P_1 \ge p_1, p_2, p_3.$$

What about Bob? This is more complicated.

- $P_m \le P_{m-1} \le \cdots \le P_3 \le P_2 \le P_1$.
- $Q_m \le Q_{m-1} \le \cdots \le Q_3 \le Q_2 \le Q_1$.
- $P_3 \le Q_{m-2}$
- Hence $P_3 \le Q_{m-2} \le Q_{m-3} \le \cdots \le Q_1$.

Assume, by way of contradiction, that Bob thinks p_1 is smaller than one of q_1, q_2, q_3. Since Bob thinks $q_1 \le q_2 \le q_3$ we can assume Bob thinks

$$\begin{array}{ll} p_1 & \le q_3 \\ \frac{P_1}{3} & \le Q_{m-2} \\ P_1 & \le 3Q_{m-2} \end{array}$$

Hence Bob thinks

$$P_1 \le Q_1 \cup Q_2 \cup Q_3.$$

and

$$P_1 \leq Q_4 \cup Q_5 \cup Q_6.$$

Since $P_2 \leq P_1$ Bob thinks

$$P_2 \leq Q_4 \cup Q_5 \cup Q_6.$$

Since $P_3 \leq Q_{m-2}$ and all P_4, P_5, $\ldots$ are even smaller, they are all smaller than Q_{m-2} and hence smaller than Q_7, Q_8, $\ldots$. Putting all of this together we get the following

$$\begin{array}{rcl} P_1 & \leq & Q_1 \cup Q_2 \cup Q_3 \\ P_2 & \leq & Q_4 \cup Q_5 \cup Q_6 \\ P_3 & \leq & Q_7 \\ P_4 & \leq & Q_8 \\ \vdots & \vdots & \vdots \\ P_{m-6} & \leq & Q_{m-2} \end{array}$$

Hence

$$P_1 \cup \cdots \cup P_{m-6} \leq Q_1 \cup \cdots \cup Q_{m-2}$$

$$P - (P_{m-5} \cup P_{m-4} \cup P_{m-3} \cup P_{m-2} \cup P_{m-1} \cup P_m) \leq Q - (Q_{m-1} \cup Q_m) \leq Q$$

Hence if Bob discards the 6 smallest pieces from P then Bob has $\leq$ what Alice has. This contradicts the definition of m. Hence Bob thinks

$$p_1 = p_2 = p_3 > q_1, q_2, q_3.$$

∎

Note 4.4 The source I am working from [1] seems to need m such that even if SEVEN (not SIX) pieces are missing Bob is still happy. Also, they seem to think that (in our terms) $q_1 = q_2 = q_3 > p_1, p_2, p_3$, but $p_1 = p_2 = p_3 \geq q_1, q_2, q_3$. Either I simplified the proof or I am missing some very subtle point. Extra Credit if you can find the point I am missing (warning- there might not be one).

Exercise 2 Let p be how much Bob values P. Let q be how much Bob values Q. Determine a good value of m as a function of p, q.

4.3 Making Alice and Bob Have an Advantage Over Each Other

We can now use this Theorem 3.2 and Lemma 4.3 to obtain a protocol where Alice and Bob have an advantage over each other!

Theorem 4.5 *Alice, Bob, Carol, and Donna are looking at 3 pieces P, Q, R (any of these could themselves be composed of many pieces). Alice thinks $P = Q$. Bob thinks $P > Q$. There is a protocol such that at the end:*

- *all but some trim T of the cake has been divided*
- *on the part that was divided the division is Envy Free*
- *Alice has an advantage over Bob*
- *Bob has an advantage over Alice.*

Proof:
Protocol ADV is in 2 phases.
ADV PHASE ONE

1. Alice and Bob perform Protocol SIX FUNKY PIECES from Lemma 4.3 on P and Q to obtain (small) pieces $p_1, p_2, p_3, q_1, q_2, q_3$ such that
   - Alice thinks $q_1 = q_2 = q_3 > p_1, p_2, p_3$.
   - Bob thinks $p_1 = p_2 = p_3 > q_1, q_2, q_3$.

   (They follow the advice from Lemma 4.3.)
2. Carol trims at most 1 of the 6 pieces (creating a tie for first). Call this trim T and put it aside.
3. Donna takes one of the 6 pieces.
4. Carol takes one of the 6 pieces. If the one she trimmed is available she must take it.

5. Bob takes one of the untrimmed p_i pieces. Note that there must be one of them, untrimmed, left since if Carol trimmed a piece then either Donna or Carol took it.

6. Alice takes one of the untrimmed q_i pieces. Note that there must be one of them, untrimmed, left since if Carol trimmed a piece then either Donna or Carol took it, AND, Bob took a p_i piece.

END OF ADV PHASE ONE

We leave it as an exercise that if any player does not follow the advice they may end up with $< 1/4$ of what was divided. We also leave it as an exercise that, of what was so far divided up (which might just be crumbs!) the division is envy-free.

Note that Alice has a q_i piece which she think is BIGGER than the puny p_i piece that Bob got.

Note that Bob has a p_i piece which she think is BIGGER than the puny q_i piece that Bob got.

We summarize and be more concrete:

- Alice has q_1 which is TINY but bigger than p_1 in her eyes. Lets say Alice thinks $q_1 - p_1 < \epsilon'$ where ϵ' is rational.

- Bob has p_1 which is TINY but bigger than q_1 in his eyes. Lets say Bob thinks $p_1 - q_1 < \epsilon''$ where ϵ'' is rational.

- Let $\epsilon = \min\{\epsilon', \epsilon''\}$.

- If Alice got ϵ more cake, Bob wouldn't mind.

- If Bob got ϵ more cake, Alice wouldn't mind.

ADV PHASE TWO

1. Alice writes down a rational number ϵ' (Alice thinks $q_1 - p_1 < \epsilon'$.)

2. Bob writes down a rational number ϵ'' (Bob thinks $p_1 - q_1 < \epsilon''$.)

3. Let $\epsilon = \min\{\epsilon', \epsilon''\}$.

4. Let P be the rest of the cake— the trim together with whatever did not go into the 6 pieces. P may be most of the cake.

5. Run Protocol ALL-HAPPY from Theorem 3.2 on P with parameter ϵ. Recall that at the end of this protocol P will be divided in an envy-free manner except for a piece T that everyone thinks is $\leq \epsilon$.

END OF ADV PHASE TWO

Since the trim is $\leq \epsilon$ to both Alice and Bob, and they both don't care if the other gets ϵ, they each have an advantage over each other.

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4.4 The Final Envy Free Protocol for 4 People

We can now present the final protocol. The KEY is that we keep a list of pairs-of-people who don't care what the other one gets from the trim. The list, called LIST, is initially empty. We call the original pie PIE. We have PIE and LIST as parameters of the input since we will call the protocol on itself.

The actual protocol will be to call the following protocol on $(CAKE, \emptyset)$.
ENVYFREE4(PIE,LIST)

1. Alice cuts PIE into TWELVE pieces (equal).
2. Bob, Carol, and Donna writes down on a piece of paper E (if they think the pieces are all equal) or N (if they think they are not all equal). Let $EQUAL$ be the set of people who think that the pieces are equal. Let $NOTEQUAL$ be the set of people who think that the pieces are not equal.
    (a) If there is a pair (of people) that is not on the LIST that disagree (one thinks EQUAL one things NOT EQUAL) then run Protocol ADV from Theorem 4.5 so that they each now have an advantage over each other. Put the pair on the LIST. Then call $ENVYFREE4(T, LIST)$. (We will see later that the recursion will bottom out.)
    (b) In this case every pair who disagree are on the list. Note that every person in $NOTEQUAL$ has an advantage over everyone in $EQUAL$. Hence the people in $NOTEQUAL$ don't care what the people in $EQUAL$ get. If $EQUAL$ has one person, he or she gets

all 12 pieces. If $EQUAL$ has 2 people, they each get 6 pieces. If $EQUAL$ has 3 people, they each get 4 pieces. If $EQUAL$ has 4 people, they each get 3 pieces.

Since there are 4 people there are at most 6 pairs. If the protocol is called with $LIST$ equal to the set of all pairs then it will not call itself, it will terminate. Each time the protocol is called one more pair is put on the list. Hence the protocol is called at most 6 times.

References

[1] S. J. Brams and A. D. Taylor. An envy-free cake division protocol. *The American Mathematical Monthly*, 102:9–18, 1995.