Excerpt from
Purely Combinatorial Proofs of Van Der Waerden-Type Theorems
by William Gasarch and Andy Parrish

Lemma 0.0.1 *For all k, s, c, there exists $U = U(k, s, c)$ such that for every c-coloring $\chi : [U] \rightarrow [c]$ there exists a, d such that*

$$\chi(a) = \chi(a + d) = \cdots = \chi(a + (k - 1)d) = \chi(sd)$$

Proof: We prove this by induction on c. Clearly, for all k, s,

$$U(k, s, 1) = \max\{k, s\}.$$

We assume $U(k, s, c - 1)$ exists and show that $U(k, s, c)$ exists. We will show that

$$U(k, s, c) \leq W((k - 1)sU(k, s, c - 1) + 1, c).$$

Let χ be a coloring of $[W((k-1)sU(k, s, c-1) + 1, c)]$. By the definition of W there exists a, d such that

$$\chi(a) = \chi(a + d) = \cdots = \chi(a + (k - 1)sU(k, s, c - 1)d).$$

Assume the color is RED. There are several cases.

Case 1: If sd is RED then since $a, a + d, \ldots, a + (k - 1)d$ are all RED, we are done.

Case 2: If $2sd$ is REDthen since. $a, a + 2d, a + 4d, \ldots, a + 2(k - 1)d$ are all RED, we are done.

$\vdots$

Case U(k,s,c-1): If $U(k, s, c - 1)sd$ is REDthen since
a,$a + U(k, s, c - 1)d$,$a + 2U(k, s, c - 1)d$,. . ., $a + (k - 1)U(k, s, c - 1)d$
are all RED, we are done.

Case U(k,s,c-1)sd+1: None of the above cases happen. Hence
$sd, 2sd, 3sd, \ldots, U(k, s, c - 1)sd$
are all NOT RED.
Consider the coloring $\chi' : [U(k, s, c - 1)] \rightarrow [c - 1]$ defined by

$$\chi'(x) = \chi(xsd).$$

The KEY is that NONE of these will be colored REDso there are only $c-1$ colors. By the inductive hypothesis there exists a', d' such that

$$\chi'(a') = \chi'(a'+d') = \cdots = \chi'(a'+(k-1)d') = \chi'(sd')$$

so

$$\chi(a'sd) = \chi(a'sd+d'sd) = \cdots = \chi(a'sd+(k-1)d'sd) = \chi(sd'sd)$$

Let $A = a'sd$ and $D = d'sd$. Then

$$\chi(A) = \chi(A+D) = \cdots = \chi(A+(k-1)D = \chi(sD).$$

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