Lemma 0.0.1 For all \( k, s, c \), there exists \( U = U(k, s, c) \) such that for every \( c \)-coloring \( \chi : [U] \to [c] \) there exists \( a, d \) such that

\[
\chi(a) = \chi(a + d) = \cdots = \chi(a + (k - 1)d) = \chi(sd)
\]

Proof: We prove this by induction on \( c \). Clearly, for all \( k, s \),

\[
U(k, s, 1) = \max\{k, s\}.
\]

We assume \( U(k, s, c - 1) \) exists and show that \( U(k, s, c) \) exists. We will show that

\[
U(k, s, c) \leq W((k - 1)sU(k, s, c - 1) + 1, c).
\]

Let \( \chi \) be a coloring of \( [W((k - 1)sU(k, s, c - 1) + 1, c)] \). By the definition of \( W \) there exists \( a, d \) such that

\[
\chi(a) = \chi(a + d) = \cdots = \chi(a + (k - 1)sU(k, s, c - 1)d).
\]

Assume the color is RED. There are several cases.

Case 1: If \( sd \) is RED then since \( a, a + d, \ldots, a + (k - 1)d \) are all RED, we are done.

Case 2: If \( 2sd \) is RED then since \( a, a + 2d, a + 4d, \ldots, a + 2(k - 1)d \) are all RED, we are done.

\vdots

Case \( U(k, s, c - 1)sd \): If \( U(k, s, c - 1)sd \) is RED then since

\[
a, a + U(k, s, c - 1)d, a + 2U(k, s, c - 1)d, \ldots, a + (k - 1)U(k, s, c - 1)d
\]

are all RED, we are done.

Case \( U(k, s, c - 1)sd + 1 \): None of the above cases happen. Hence

\[
sd, 2sd, 3sd, \ldots, U(k, s, c - 1)sd
\]

are all NOT RED.

Consider the coloring \( \chi' : [U(k, s, c - 1)] \to [c - 1] \) defined by

\[
\chi'(x) = \chi(xs)\text{.}
\]
The KEY is that NONE of these will be colored RED so there are only \( c - 1 \) colors. By the inductive hypothesis there exists \( a', d' \) such that 

\[
\chi'(a') = \chi'(a' + d') = \cdots = \chi'(a' + (k - 1)d') = \chi'(sd')
\]

so

\[
\chi(a'sd) = \chi(a'sd + d'sd) = \cdots = \chi(a'sd + (k - 1)d'sd) = \chi(sd'sd)
\]

Let \( A = a'sd \) and \( D = d'sd \). Then

\[
\chi(A) = \chi(A + D) = \cdots = \chi(A + (k - 1)D) = \chi(sD).
\]