(From SIGACT NEWS Volume 41, Vol 2, 2010)

Review of[1]
Analytic Combinatorics
by Philippe Flajolet and Robert Sedgewick
Published by Cambridge Press, 2009
824 pages, Hardcover
Amazon: \$66.00 new, \$81.00 used[2]

Review by Miklós Bóna

1 Introduction

This is a very important book, and we will say more about its importance at the end. The goal of the authors is to explain how to use the techniques of real and complex analysis in order to enumerate combinatorial objects.

2 Summary

The book consists of three parts. The first part, which is a necessary prerequisite for the other parts, is about Symbolic Methods. Most combinatorial structures can be put together from simpler parts according to certain rules. For instance, set partitions are just (unordered) sets of blocks, permutations are sets of cycles, compositions are sequences of positive integers, and integer partitions are multisets of positive integers. The authors call structures that can be built up like this *admissible*, and introduce symbolic notation to describe the way in which they are built up. For instance

$$\mathcal{P} = \mathcal{SET}(\mathcal{CYC})$$

means that the class $\mathcal{P}$ of permutations of an n-element set are nothing else but a set of disjoint cycles on that set so that each element of the set is part of a cycle, while

$$\mathcal{A} = \mathcal{SEQ}(\mathcal{CYC})$$

means that the class $\mathcal{A}$ of alignments is equal to the class of *linearly ordered* set (or sequence) of cycles on the same set.

Then the authors explain how symbolic equations can be turned into equations for the generating functions of the classes that they describe. Chapter I is devoted to unlabeled structures and the ordinary generating functions enumerating their classes, while Chapter II treats labeled structures and their exponential generating functions. The Lagrange Inversion Formula is also covered.

Chapter III takes the discussion to a higher level by introducing bivariate generating functions. It is explained how to use these generating functions to compute the average value of a certain parameter of a combinatorial structure. This technique can be used to compute the average number of cycles in a randomly selected alignment of size n, or the average number of parts in a randomly selected composition of n whose parts have to be in a certain set, and so on. The main idea is that

[2] This is not a typo. Someone thinks its worth more used then new.

if $A(x,u)=\sum_{n,k} a(n,k)x^n u^k$, where $a(n,k)$ is the number of objects of size n in which the value of a certain parameter is k, then the coefficient of x^n in

$$\frac{d}{du}A(x,u)|_{u=1}$$

is the *cumulative value* of that parameter for all our objects of size n. Hence, dividing that the number by $[x^n]A(x,1)$, that is, by the total number of our objects of size n, we get the average value of the parameter. Taking higher derivatives of the generating function can lead to the computation of higher moments of the parameter.

A spectacular application is that the random composition of the integer n contains about $n/4$ parts equal to 1, about $n/8$ parts equal to 2, and so on, $\frac{n}{2^{i+1}}$ parts equal to i. In particular, the if $i > (\log_2 n) - 1$, then the typical composition of n will not contain a part equal to i. The result describing how the average composition looks like is called the *profile* of the composition. We later find results on the profiles of other objects as well, such as permutations and various kinds of graphs.

The second, and most extensive part of the book is on Asymptotic Computation. This is the subject of Chapters IV through VIII. Some knowledge of complex analysis is useful for this chapters, but it is not necessary; most of the needed techniques can be learned from this book as well. First, we are explained the notion of analytic functions and their singularities, with the fundamental theorem that the exponential order of the coefficients of an analytic function f is equal to $1/\rho$, where ρ is the absolute value of the singularity ρ of f that is closest to 0. For rational functions, which are the ratio of two polynomials, and for meromorphic functions, which are the ratio of two analytic functions, more precise results are given.

We then move on to *Singularity Analysis*, which is perhaps the most unique part of the book. It is explained that when looking for the growth of the coefficients of a power series, it is not simply the location of the singularities that matters, but also their *number* (there can be several singularities on the same circle around 0), and their *type*. For instance, while $f(x)=\frac{1}{1-x}$, $g(x)=1/\sqrt{1-x}$ and $h(x)=\log(1/(1-x))$ all have a unique singularity at $x=1$, the growth rates of their coefficients are different in their subexponential terms. In particular, $f_n=1$, $g_n \sim \frac{1}{\sqrt{\pi n}}$, and $h_n=1/n$. The authors show that while the location of the dominant singularity determines the exponential order of the coefficients, the type and number of these singularities determines the subexponential factors. The three examples above are representative to the three types of singularities treated here, namely poles, square-root type, and logarithmic type singularities. We again find some spectacular and surprising applications besides the standard ones.

A large family of applications concerns combinatorial classes that are enumerated by a generating function $T(x)$ that satisfies an equation of the type $T(x)=x\phi(T(x))$, for some non-linear function ϕ. Classes of rooted trees for which this holds are called *simple tree varieties*. Several sections of Chapter VII are devoted to exploring the parameters of these varieties. For instance, if τ is the unique positive real root of the equation $\phi(x)=x\phi'(x)$, then it is shown that the radius of convergence $T(x)$ is $\rho=\tau/\phi(\tau))$. If n, the number of vertices of the trees, goes to infinity, then the probability generating function of the degree of the root is $u\phi'(\tau u)/\phi'(\tau)$. For example, if our tree variety is that of unary-binary unlabeled plane trees, (each non-leaf vertex has one or two children), then $\phi(w)=1+w+w^2$, so $\tau=1$, and $\rho=1/3$. The mentioned probability generating function is $u/3+2u^2/3$, meaning that for large n, in about two-third of all unary-binary trees on n vertices, the root will have two children. Several other parameters, such as the average number of vertices at distance k from the root, the average number of vertices of degree k, and the average path length can easily be computed once ϕ and τ are known.

Chapter VIII takes the discussion one level higher by covering the *Saddle Point Method*. This is the only part of the book where familiarity with complex integrals, at least to the level of knowing Cauchy's coefficient formula,

$$[z^n]G(z) = \frac{1}{2i\pi}\int_C G(z)\frac{dz}{z^{n+1}}$$

is necessary to understand the essence of the method (if not, strictly speaking, to be able to use the method in simple cases). Here C is a contour that encircles the origin, lies within the domain where G is analytic, and is positively oriented.

"Saddle points" are points where the derivative of a function is zero, while the function itself is not. The saddle point method is to be applied when singularity analysis fails, either because the function at hand has no singularities, such as $e^{x+x^2/2}$ (the exponential generating function of involutions), or when the function has singularities not covered by the other methods of singularity analysis, such as $e^{1/(1-x)}$ (the exponential generating function of broken permutations. The analytic basis of the method is selecting a contour C that goes through or very near a saddle point of G, and then applying Cauchy's coefficient formula, displayed above. Besides the two mentioned examples, the method is applied to set partitions, various classes of permutations, and many other admissible structures (structures that can be built up by composing sets, lines, and cycles).

Finally, Chapter IX, which constitutes the third, shortest part of the book, is on the probabilistic aspects of analytic combinatorics. Given a combinatorial class and a parameter, such as permutations and their number of cycles, one can ask what the distribution of the parameter is when the objects are of size n. Then one can ask what happens to that distribution as n goes to infinity. Various notions of convergence are discussed for these distributions. Then the authors show how to use multivariate generating functions to prove results about the limits of sequences of distributions. In the mentioned example, as n goes to infinity, the distribution of the number of cycles of permutations of size n converges to a normal distribution.

The book concludes with three appendices, on Elementary Combinatorics, Complex Analysis, and Probability, which make the book self-contained.

3 Opinion

This reviewer has been teaching a graduate class from this book this semester. It is a serious time commitment. The book has enough material for three or more semesters if all details are covered, so the instructor faces very serious choices as to which topics to include. As the discussion advances, the part of analysis grows at the expense of combinatorics, so it is the instructor's task to keep balance. Students in the class like the class and the book in general, though they disliked the fact that Exercises are not clearly marked in the book (there are many examples where filling in the details is left as an exercise, but the difficulty level of these was often not clear for the students at the outset).

However, these are minor points of criticism, and most books that are meant both for students and researchers will have a few minor problems like that. More importantly, the book does nothing less than it *creates* the notion of Analytic Combinatorics. Before this book, when someone said "Analytic Combinatorics", it was not clear what he meant by it. There was no wide consensus on what works belong there, unlike in Algebraic Combinatorics, or Probabilistic Combinatorics. The Online Journal of Analytic Combinatorics was only started a few years ago. Because of the breadth, and depth of topical coverage, the highly applicable results and the enjoyable writing that characterize this book, Analytic Combinatorics is now defined. The authors wrote the book on it.