A Problem Rejected from the UMCP Math Comp in 2011

by Bill Gasarch

The following problem was rejected from the UMCP Math Competition in 2011 because it was thought to be too hard. I came up with it by knowing a general theorem and working backwards to a problem, so I can't tell how hard it is. Those who thought it was too hard are very sharp so I am inclined to agree with them.

However, here is the problem and several solutions for your amusement. The solutions are by me (from something I read), David Eppstein, Matt Howell, and Anonymous.

PROBLEM: Prove or disprove: there exist natural numbers $x_1, \ldots, x_{10}$ such that

- $2011 = x_1 + \cdots + x_{10}$, and
- $1 = \frac{1}{x_1} + \cdots + \frac{1}{x_{10}}$.

MY SOLUTION BASED ON READING ABOUT THEOREMS LIKE THIS: We call a number that can be written as the sum of m numbers such that the sum of the reciprocals of those numbers sums to 1 *m-cool*.

Lemma: If x is n-cool then (a) $20x+11$ is $n+3$-cool, (b) $2x+8$ is $n+2$-cool, and (c) $4x+6$ is $n+2$-cool.

Proof: We assume $x = \sum_{i=1}^{n} x_i$ where $\sum_{i=1}^{n} \frac{1}{x_i} = 1$

(a) $20x+11 = 2+4+5+\sum_{i=1}^{n} 20x_i$ and $\frac{1}{2}+\frac{1}{4}+\frac{1}{5}+\sum_{i=1}^{n} \frac{1}{20x_i} = \frac{3}{4}+\frac{1}{5}+\frac{1}{20} = 1$.

(b) $2x+8 = 4+4+\sum_{i=1}^{n} 2x_i$ and $\frac{1}{4}+\frac{1}{4}+\sum_{i=1}^{n} \frac{1}{2x_i} = \frac{1}{2}+\frac{1}{2} = 1$.

(c) $4x+6 = 2+4+\sum_{i=1}^{n} 4x_i$ and $\frac{1}{2}+\frac{1}{4}+\sum_{i=1}^{n} \frac{1}{4x_i} = \frac{3}{4}+\frac{1}{4} = 1$

End of Proof

Let *cool*(x) be a value n such that x is n-cool.

$2011 = 20 \times 100 + 11$. So $cool(2011) = cool(100) + 3$.

$100 = 2\times 46+8$. So $cool(100) = cool(46)+2$. So $cool(2011) = cool(46)+5$.

$46 = 4\times 10+6$. So $cool(46) = cool(10)+2$. So $cool(2011) = cool(10)+7$.

$10 = 4+4+2$ and $\frac{1}{4}+\frac{1}{4}+\frac{1}{2} = 1$. So 10 is 3-cool. So 2011 is 10-cool.

WHAT ARE $x_1, \ldots, x_{10}$? The proof is entirely constructive so we use it to find the numbers. (Note: the solution to the problem does not require this.)

$10 = 4+4+2$

$46 = 2+4+4\times 10 = 2+4+4(4+4+2) = 2+4+16+16+8 = 16+16+8+4+2$

$$100 = 4+4+2\times 46 = 4+4+2(16+16+8+4+2) = 4+4+32+32+16+8+4$$
$$= 32+32+16+8+4+4+4$$

$$2011 = 2+4+5+20\times 100 = 2+4+5+20(32+32+16+8+4+4+4)$$
$$= 2+4+5+640+640+320+160+80+80+80$$
$$= 640+640+320+160+80+80+80+5+4+2$$

Note that

$$\left(\frac{1}{640}+\frac{1}{640}+\frac{1}{320}+\frac{1}{160}+\frac{1}{80}\right)+\left(\frac{1}{80}+\frac{1}{80}\right)+\frac{1}{5}+\frac{1}{4}+\frac{1}{2}=$$

$$\frac{1}{40}+\frac{1}{40}+\frac{1}{5}+\frac{1}{4}+\frac{1}{2}=$$

$$\frac{1}{20}+\frac{1}{5}+\frac{1}{4}+\frac{1}{2}=$$

$$\frac{1}{4}+\frac{1}{4}+\frac{1}{2}=1$$

Note: Ronald Graham proved that all numbers $n \ge 78$ are cool in *A Theorem on Partitions*, Journal of the Aust. Math Society, 1963. The following can be used to prove it (I do not think this was the original proof- the idea for it came from a comment on my blog)
Claim: If x is n-cool then $2x+8$, $2x+9$, $3x+6$, $3x+8$ are $n+2$-cool.
Proof of Claim:
We assume $x = \sum_{i=1}^n x_i$ where $\sum_{i=1}^n \frac{1}{x_i} = 1$

1. $2x+8 = 8+\sum_{i=1}^n 2x_i = 4+4+\sum_{i=1}^n 2x_i$. Note that $\frac{1}{4}+\frac{1}{4}+\sum_{i=1}^n \frac{1}{2x_i} = \frac{1}{2}+\frac{1}{2} = 1$.
2. $2x+9 = 9+\sum_{i=1}^n 2x_i = 3+6+\sum_{i=1}^n 2x_i$. Note that $\frac{1}{3}+\frac{1}{6}+\sum_{i=1}^n \frac{1}{2x_i} = \frac{1}{2}+\frac{1}{2} = 1$.
3. $3x+6 = 6+\sum_{i=1}^n 3x_i = 3+3+\sum_{i=1}^n 3x_i$. Note that $\frac{1}{3}+\frac{1}{3}+\sum_{i=1}^n \frac{1}{3x_i} = \frac{2}{3}+\frac{1}{3} = 1$.

4. $3x+8 = 8+\sum_{i=1}^{n} 3x_i = 2+6+\sum_{i=1}^{n} 3x_i$. Note that $\frac{1}{2}+\frac{1}{6}+\sum_{i=1}^{n} \frac{1}{3x_i} = \frac{2}{3}+\frac{1}{3} = 1$.

End of Proof of Claim

We probably don't even need the last two. Anyway, with this Claim and some base cases, one can prove Graham's theorem mentioned in the note.

SOLUTION BY A HIGH SCHOOL SENIOR SAM ZBARSKY: I asked Sam Zbarsky, a High School student who has done very well in the UMCP Math Competitions, to look at it. He got it in 30 minutes. Here is his thought process and solution:

I noticed that if I include $2, 4, 5$ then I need 7 numbers that sum to 2000 and reciprocals sum to $\frac{1}{20}$. If all are divisible by 20 then the problem is

$$20y_1 + \cdots + 20y_7 = 2000$$

$$\frac{1}{20y_1} + \cdots + \frac{1}{20y_7} = \frac{1}{20}$$

which can be expressed as

$$y_1 + \cdots + y_7 = 100$$

$$\frac{1}{y_1} + \cdots + \frac{1}{y_7} = 1$$

Then I guessed and checked until I got $2, 6, 8, 15, 15, 24, 30$. Hence the final solution is: $2, 4, 5, 40, 120, 160, 300, 300, 480, 600$.

SOLUTION BY DAVID EPPSTEIN:

David Eppstein emailed me this solution after reading about the problem on my blog. I filled in some of the easy missing steps and the result is below. Its written as though its all from him.

After extensive hand calculation (no computer) I found the solution

$$3, 4, 7, 16, 16, 16, 20, 43, 80, 1806.$$

My method was to start with the equation

$$\frac{1}{2}+\frac{1}{3}+\frac{1}{7}+\frac{1}{43}+\frac{1}{1806} = 1$$

$$2 + 3 + 7 + 43 + 1806 = 1861$$

coming from Sylvester's sequence, and try to modify it to make its number of terms larger and the difference from 2011 smaller. These five terms have sum 1861, off by 150, and I noticed that if I replaced the $\frac{1}{2}$ term by $\frac{1}{4} + \frac{1}{4}$ I could make the difference an even rounder number, 144. So now I have

$$\frac{1}{3} + \frac{1}{4} + \frac{1}{4} + \frac{1}{7} + \frac{1}{43} + \frac{1}{1806} = 1$$

$$3 + 4 + 4 + 7 + 43 + 1806 = 1867$$

I observed (along with several similar observations) that if I could find five integers whose reciprocals summed to 1 and whose sum was 37, then I could use that solution to replace one of the $\frac{1}{4}$ terms to get a solution to the whole problem. (We'll see how later.)

I then built a big table of all the sets of four unit fractions summing to one that I could think of, and computed the difference of the sum of denominators of each with 37. When I found that

$$2 + 4 + 5 + 20 = 31 \text{ which differs from 37 by 6}$$

$$\frac{1}{2} + \frac{1}{4} + \frac{1}{5} + \frac{1}{20} = 1$$

I realized that I could substitute $\frac{1}{2} = \frac{1}{4} + \frac{1}{4}$ to get

$$4 + 4 + 4 + 5 + 20 = 37$$

$$\frac{1}{4} + \frac{1}{4} + \frac{1}{4} + \frac{1}{5} + \frac{1}{20} = 1$$

Multiply all of the numbers by 4 to get

$$16 + 16 + 16 + 20 + 80 = 148$$

$$\frac{1}{16} + \frac{1}{16} + \frac{1}{16} + \frac{1}{20} + \frac{1}{80} = \frac{1}{4}$$

In

$$\frac{1}{3}+\frac{1}{4}+\frac{1}{4}+\frac{1}{7}+\frac{1}{43}+\frac{1}{1806}=1$$

$$3+4+4+7+43+1806=1867$$

replace 4 by 16,16,16,20,80 to get

$$\frac{1}{3}+\frac{1}{4}+\frac{1}{7}+\frac{1}{16}+\frac{1}{16}+\frac{1}{16}+\frac{1}{20}+\frac{1}{43}+\frac{1}{80}+\frac{1}{1806}=1$$

$$3+4+7+16+16+16+20+43+80+1806=2011$$

Plugging that back in to my earlier calculations gave the solution, which I then verified on the computer.

Note from Bill: Would a High School Student know Sylvester's sequence? I doubt it. However, I can see a bright HS student coming up with this solution in a way similar to how David Eppstein did.

SOLUTION BY MATT HOWELLS: Matt Howells emailed me this after reading about the problem on my blog.

Here is one such assignment:

2, 4, 5, 50, 100, 100, 250, 500, 500, 500

My method was crude but somewhat systematic, I simply looked for ways to make a sum to 11 with 3 numbers, then found a way to complete the sum of the second equation up to .99 then this restricted the number of options for the last few assignments to a manageable handfull of options.

I am sure the above solution is not unique, and this problem would be a great challenge for a HS student. I'd have voted to keep it on the list for the competition.

Finally, it would be a *really* hard problem to determine the exact number of solutions to the given puzzle. Any thoughts?

SOLUTION BY ANONYMOUS:

$$2011=6+6+10+10+12+15+15+15+62+1860$$

This is one of the solutions There is also a simpler solution with 8 entries

$$2011=2+12+12+15+24+24+62+1860$$

(Note from Bill: I don't call this simpler- I call this a different problem.)

We can work these from

$$1957 = 2 + 6 + 12 + 15 + 62 + 1860$$

We miss 54, but have 4 extra entries. We can expand or change lower terms. For instance, if we expand 1/6 to 3 1/18, we gain 50 with 2 extra to go. So we can get 16 by expanding 2 to 3 1/6, so we get 2023. Now we may only expand lowest terms, expanding 12 once to two 1/24 gives us 36 gain so again with expanding 2 to 4 1/8 we get gain 66, again 2023.

Now 1/2+1/6=2/3, so we need to represent 62 with six entries but sum of reciprocials 2/3. Now suppose we want to represent 63 this is 3×21. This is not a problem - 10 is represented by 2 4 4 and 11 by 2 3 6. We thus get

$$2012 = 6 + 6 + 9 + 12 + 12 + 12 + 15 + 18 + 62 + 1860$$

By the same method we can get 60, which gives rise to 2009

$$2009 = 6 + 6 + 12 + 12 + 12 + 12 + 12 + 15 + 62 + 1860$$

We can just as easily get 2006, 2003 and 2015.

To get other years in the near past or future, from 1957, we may want to use split 1/2+1/6. Now 62 for instance is divisible by 2, so we want 28 expressed in terms of 5 numbers. 5=3+2, so lets see - we will write 28=20+8 and use 1=1/2+1/2, and use 3 representation of 10 and 2 representation of 4, which is to say 10=2+4+4 and 4=2+2, and hence 28= 4+4+4+8+8, and 62=6+8+8+8+16+16, and so another solution for the dying year:

$$2011 = 6 + 8 + 8 + 8 + 12 + 15 + 16 + 16 + 62 + 1860.$$

(Note from Bill: You didn't quite say how you got this, but I can certainly believe that you can get it from the techniques you've mentioned.)

Is it hard for HS student? Yes, in the time frame given, it is hard. Also, since there is guessing involved, it might be unfair - it takes a lot of time to think this through properly, and students would be guessing randomly.

SOLUTION BY MIKE ROMAN: Mike Roman emailed me this after seeing my blog:

I solved your blog problem, here's my answer:

$$(5, 5, 6, 8, 8, 12, 15, 32, 960, 960).$$

I arrived at these numbers as follows:

1. Noticed that 30*32 = 960 which is around half of 2011
2. Assumed 3 of my fractions would be 1/32, 1/960, 1/960
3. Used trial and error to get the other 7 fractions to add up to 29/30 and also have the denominators add up to 59 in order to satisfy both equations

Note:

- My Solution: 2,4,5,80,80,80,160,320,640,640
- Sam Solution: 2,4,5,40,120,160,300,300,480,600
- David Eppstein Solution: 3,4,7,16,16,16,20,43,80,1806
- Matt Howell Solution: 2,4,5,50,100,100,250,500,500,500
- Mike Roman Solution: 5,5,6,8,8,12,15,32,960,960
- Anonymous Solution: 6,6,10,10,12,15,15,15,62,1860
- Anonymous Solution: 6,8,8,8,12,15,16,16,62,1860

I wonder how many solutions there are. I wonder if there are any that have distinct numbers. I wonder about the computational complexity of the following problems:
Problem 1: Given $a, b \in N$ does there exist $x_1, \ldots, x_a$ such that

- $x_1 + \cdots + x_a = b$, and
- $\frac{1}{x_1} + \cdots + \frac{1}{x_a} = 1$.

Can also ask with the stipulation that the x_i's are distinct.
Problem 2: Given $a, b \in N$ and $r \in Q$, does there exist $x_1, \ldots, x_a$ such that

- $x_1 + \cdots + x_a = b$, and
- $\frac{1}{x_1} + \cdots + \frac{1}{x_a} = r$.

Can also ask with the stipulation that the x_i's are distinct.