An Exposition of How GKP Solved the Change Problem

William Gasarch [*]

Univ. of MD at College Park

1 Introduction

In Graham-Knuth-Patashnik book *Concrete Mathematics* they get a closed form for some cases of the following problem:

Given n, how many ways are there to make n cents change with pennies, nickels, dimes, quarters, and half-dollars.

We give an exposition of what they did, but shorten it to just pennies, nickels, dimes, and quarters. The Once you see this you can easily do your the pennies,nickels, dimes, quarters, half-dollars case if you wish. We then give a generalization of what they did which yields, for any finite set of coins, a formula. (One can debate if its really a formula.)

Lemma 1.1

$$\frac{1}{(1-x)^{k+1}} = \sum_{i=0}^{\infty} \binom{i+k}{k} x^i.$$

2 Pennies, Nickels, Dimes, Quarters

It is easy to see that the number of ways to make change of n cents using pennies, nickels, dimes, and quarters is the coefficient of z^n in the power series for

[*]University of Maryland, College Park, MD 20742, gasarch@cs.umd.edu

$$C(z) = \frac{1}{(1-z)(1-z^5)(1-z^{10})(1-z^{25})} = \frac{1+z+z^2+z^3+z^4}{(1-z^5)(1-z^5)(1-z^{10})(1-z^{25})}.$$

This motivates the following definition

$$\hat{C}(z) = \frac{1}{(1-z)(1-z)(1-z^2)(1-z^5)}$$

So we now have

$$C(z) = (1+z+z^2+z^3+z^4)\hat{C}(z^5)$$

We need the coefficients of $\hat{C}(z)$. Note that

$(1-z^{10}) = (1-z)(1+z+z^2+z^3+z^4+z^5+z^6+z^7+z^8+z^9)$

$(1-z^{10}) = (1-z^2)(1+z^2+z^4+z^6+z^8)$

$(1-z^{10}) = (1-z^5)(1+z^5)$

Hence

$$\hat{C}(z) = \frac{(1+z+z^2+z^3+z^4+z^5+z^6+z^7+z^8+z^9)^2(1+z^2+z^4+z^6+z^8)(1+z^5)}{(1-z^{10})^4}$$

We denote the numerator by $A(z)$ and we will denote the coefficient of z_i in $A(z)$ by a_i. $A(z)$ is easily seen to be

$$1+2z+4z^2+6z^3+9z^4+13z^5+18z^6+24z^7+31z^8+39z^9+45z^{10}+52z^{11}+$$

$$57z^{12}+63z^{13}+67z^{14}+69z^{15}+69z^{16}+67z^{17}+63z^{18}+57z^{19}+52z^{20}+45z^{21}+39z^{22}+31z^{23}+$$

$$24z^{24}+18z^{25}+13z^{26}+9z^{27}+6z^{28}+4z^{29}+2z^{30}+z^{31}$$

By Lemma 1.1

$$\frac{1}{(1-z^{10})^4}=\sum_{i=1}^{\infty}\binom{i+3}{3}z^{10i}.$$

Hence

$$\hat{C}(z^5)=A(z^5)\sum_{i=1}^{\infty}\binom{i+3}{3}z^{50i}.$$

If $n=50q+r$ where $0\le r\le 49$ then the coefficient of z^n is

$$a_r\binom{q+3}{3}+a_{r+10}\binom{q+2}{3}+a_{r+20}\binom{q+1}{3}+a_{r+30}\binom{q}{3}$$

Let $b_n=a_r\binom{q+3}{3}+a_{r+10}\binom{q+2}{3}+a_{r+20}\binom{q+1}{3}+a_{r+30}\binom{q}{3}$.

Recall that

$$C(z)=(1+z+z^2+z^3+z^4)A(z^5)\hat{C}(z^5).$$

We seek the coefficient of z^n in $C(z)$.

This is clearly

$$b_n+b_{n-1}+b_{n-2}+b_{n-3}+b_{n-4}$$

To be more explicit we can say that if $n = 50q + r$ where $0 \leq r \leq 49$ then the number of ways to make change of n cents has four cases

1. If $3 \leq r \leq 49$ then the answer is

$$a_r \binom{q+3}{3} + a_{r+10} \binom{q+2}{3} + a_{r+20} \binom{q+1}{3} + a_{r+30} \binom{q}{3}$$

$$+ a_{r-1} \binom{q+3}{3} + a_{r+9} \binom{q+2}{3} + a_{r+19} \binom{q+1}{3} + a_{r+29} \binom{q}{3}$$

$$+ a_{r-2} \binom{q+3}{3} + a_{r+8} \binom{q+2}{3} + a_{r+18} \binom{q+1}{3} + a_{r+28} \binom{q}{3}$$

$$+ a_{r-3} \binom{q+3}{3} + a_{r+7} \binom{q+2}{3} + a_{r+17} \binom{q+1}{3} + a_{r+27} \binom{q}{3}$$

2. If $r = 2$ then the answer is

$$a_2 \binom{q+3}{3} + a_{12} \binom{q+2}{3} + a_{22} \binom{q+1}{3} + a_{32} \binom{q}{3}$$

$$+ a_1 \binom{q+3}{3} + a_{11} \binom{q+2}{3} + a_{21} \binom{q+1}{3} + a_{31} \binom{q}{3}$$

$$+ a_0 \binom{q+3}{3} + a_{10} \binom{q+2}{3} + a_{20} \binom{q+1}{3} + a_{30} \binom{q}{3}$$

$$+ a_{30} \binom{q+2}{3}$$

3. If $r = 1$ then the answer is

$$a_1\binom{q+3}{3} + a_{11}\binom{q+2}{3} + a_{21}\binom{q+1}{3} + a_{31}\binom{q}{3}$$

$$+a_0\binom{q+3}{3} + a_{10}\binom{q+2}{3} + a_{20}\binom{q+1}{3} + a_{30}\binom{q}{3}$$

$$+a_{30}\binom{q+2}{3} + a_{29}\binom{q+2}{3}$$

4. If $r = 0$ then the answer is

$$a_0\binom{q+3}{3} + a_{10}\binom{q+2}{3} + a_{20}\binom{q+1}{3} + a_{30}\binom{q}{3}+$$

$$+a_{30}\binom{q+2}{3} + a_{29}\binom{q+2}{3} + a_{28}\binom{q+2}{3}$$

Let us contrast this to the result I got by using recurrences.

Theorem 2.1 *Let $n = 5(5L+M)+L_0$ where $0 \le M \le r-1$, $0 \le L_0 \le 4$, and $L \ge 1$. (So $L = \lfloor \frac{n}{25} \rfloor$, $M = \lfloor \frac{n \bmod 25}{5} \rfloor$, and $n \equiv L_0 \pmod 5$).)*

The number of ways to make n cents change using pennies, nickels, dimes, and quarters is

$$\frac{1}{24}\Big((L+1)(50L^2+(8530M)L+6M^2+24M+24)\Big)$$

$$+\frac{2L+(1+(-1)^L)(1+(-1)^{M+1})+(1+(-1)^{L+1})}{16}.$$

3 Generalized

Theorem 3.1 *Let $\{t_1 < \cdots < t_v\}$ be a set of natural numbers. Then the number of ways to make change of n cents using this set of coins is*

Proof: It is easy to see that the number of ways to make change of n cents using the coins in C is the coefficient of z^n in

$$C(z) = \prod_{i=1}^{v} \frac{1}{(1 - z^{t_i})}$$

Let t be the least common multiple of $\{t_1, \ldots, t_v\}$. For $1 \le i \le v$ let f_i be the polynomial such that $(1 - z^t) = (1 - z^{t_i}) f_i(z)$. Then

$$C(z) = \frac{f_1(z) \cdots f_v(z)}{(1 - z^t)^v}$$

Let $A(z) = f_1(z) \cdots f_v(z)$. Let M be the degree of $A(z)$ which is $((t - t_1) + (t - t_2) + \cdots + (t - t_v))$. Let the coefficient of z^j in $A(z)$ be a_j. By Lemma 1.1

$$C(z) = \left(\sum_{j=0}^{M} a_j z^j \right) \left(\sum_{i=0}^{\infty} \binom{i+v-1}{v-1} z^{ti} \right) = \sum_{j=0}^{M} \sum_{i=0}^{\infty} a_j \binom{i+v-1}{v-1} z^{ti+j}.$$

The coefficient of z^n is

$$\sum_{0 \le j \le M : j \equiv n \bmod t} a_j \binom{\frac{n-j}{t} + v - 1}{v-1}$$

∎