Large Ramsey Theorem

An Exposition by Bill Gasarch

The Infinite Ramsey Theorem was proven by Ramsey [2]. The Large Ramsey Theorem was proven by Paris and Harrington [1]. The Large Ramsey Theorem is, in and of itself, not hard to prove. (That might be hindsight talking.) The main contribution of Paris and Harrington was showing that the Large Ramsey Theorem was ind of PA. Alas, we do not prove that here.

1 Infinite Ramsey Theorem

Notation 1.1 K_{N} is the graph (V, E) where

$$\begin{aligned} V = &\ \mathsf{N} \\ E = &\ \{\{x, y\} \mid x, y \in \mathsf{N}\} \end{aligned}$$

Def 1.2 Let $G = (V, E)$ be a graph with $V = \mathsf{N}$, and let COL be a coloring of the edges of G. A set of edges of G is *monochromatic* if they are all the same color (this is the same as for a finite graph).

G has a monochromatic K_{N} if there is an infinite set V' of vertices (in V) such that

- there is an edge between every pair of vertices in V'
- all the edges between vertices in V' are the same color

Theorem 1.3 *Every 2-coloring of the edges of K_{N} has a monochromatic K_{N}.*

Proof:

Let COL be a 2-coloring of K_{N}. We define an infinite sequence of vertices,

$$x_1, x_2, \ldots,$$

and an infinite sequence of sets of vertices,

$$V_0, V_1, V_2, \ldots,$$

that are based on COL.

Here is the intuition: Vertex $x_1 = 1$ has an infinite number of edges coming out of it. Some are RED, and some are BLUE. Hence there are an infinite number of RED edges coming out of x_1, or there are an infinite number of BLUE edges coming out of x_1 (or both). Let c_1 be a color such that x_1 has an infinite number of edges coming out of it that are colored c_1. Let V_1 be the set of vertices v such that $COL(\{v, x_1\}) = c_1$. Then keep iterating this process.

We now describe it formally.

$$
\begin{aligned}
V_0 &= \mathsf{N} \\
x_1 &= 1
\end{aligned}
$$

$$
c_1 = \begin{cases} \text{RED} & \text{if } |\{v \in V_0 \mid COL(\{v, x_1\}) = \text{RED}\}| \text{ is infinite} \\ \text{BLUE} & \text{otherwise} \end{cases}
$$

$$
V_1 = \{v \in V_0 \mid COL(\{v, x_1\}) = c_1\} \text{ (note that } |V_1| \text{ is infinite)}
$$

Let $i \geq 2$, and assume that V_{i-1} is defined. We define x_i, c_i, and V_i:

$$
x_i = \text{ the least number in } V_{i-1}
$$

$$
c_i = \begin{cases} \text{RED} & \text{if } |\{v \in V_{i-1} \mid COL(\{v, x_i\}) = \text{RED}\}| \text{ is infinite} \\ \text{BLUE} & \text{otherwise} \end{cases}
$$

$$
V_i = \{v \in V_{i-1} \mid COL(\{v, x_i\}) = c_i\} \text{ (note that } |V_i| \text{ is infinite)}
$$

How long can this sequence go on for? Well, x_i can be defined if V_{i-1} is nonempty. We an show by induction that, for every i, V_i is infinite. Hence the sequence

$$
x_1, x_2, \ldots,
$$

is infinite.

Consider the infinite sequence

$$
c_1, c_2, \ldots
$$

Each of the colors in this sequence is either RED or BLUE. Hence there must be an infinite sequence $i_1, i_2, \ldots$ such that $i_1 < i_2 < \cdots$ and

$$
c_{i_1} = c_{i_2} = \cdots
$$

Denote this color by c, and consider the vertices

$$x_{i_1}, x_{i_2}, \cdots$$

It is easy to see that these vertices form a monochromatic K_{N}. ∎

2 Proof of Large Ramsey Theorem

(This is due to Paris and Harrington.)

Usually the labels on the vertices did *not* matter. For this result they *do* matter.

Def 2.1 A finite set $F \subseteq \mathsf{N}$ is called *large* if the size of F is at least as large as the smallest element of F.

Example 2.2

1. The set $\{1, 2, 10\}$ is large: It has 3 elements, the smallest element is 1, and $3 \geq 1$.
2. The set $\{5, 10, 12, 17, 20\}$ is large: It has 5 elements, the smallest element is 5, and $5 \geq 5$.
3. The set $\{20, 30, 40, 50, 60, 70, 80, 90, 100\}$ is not large: It has 9 elements, the smallest element is 20, and $9 < 20$.
4. The set $\{5, 30, 40, 50, 60, 70, 80, 90, 100\}$ is large: It has 9 elements, the smallest element is 5, and $9 \geq 5$.
5. The set $\{101, \ldots, 190\}$ is not large: It has 90 elements, the smallest element is 101, and $90 < 101$.

We will be considering monochromatic K_m's where the underlying set of vertices is a large set. We need a definition to identify the underlying set.

Def 2.3 Let COL be a 2-coloring of K_n. A set A of vertices is *homogeneous* if there exists a color c such that, for all $x, y \in A$ with $x \neq y$, $COL(\{x, y\}) = c$. In other words, all of the edges between elements of A are the same color. One could also say that there is a monochromatic $K_{|A|}$.

Let COL be a 2-coloring of K_n. Recall that the vertex set of K_n is $\{1, 2, \ldots, n\}$. Consider the set $\{1, 2\}$. It is clearly both homogeneous and large (using our definition of large). Hence the statement

"for every $n \geq 2$, every 2-coloring of K_n has a large homogeneous set"

is true but trivial.

What if we used $V = \{m, m+1, \ldots, m+n\}$ as our vertex set? Then a large homogeneous set would have to have size at least m.

Notation 2.4 K_n^m is the graph with vertex set $\{m, m+1, \ldots, m+n\}$ and edge set consisting of all unordered pairs of vertices. The superscript (m) indicates that we are labeling our vertices starting with m, and the subscript (n) is one less than the number of vertices.

Note 2.5 The vertex set of K_n^m (namely, $\{m, m+1, \ldots, m+n\}$) has $n+1$ elements. Hence if K_n^m has a large homogeneous set, then $n+1 \geq m$ (equivalently, $n \geq m-1$). We could have chosen to use K_n^m to denote the graph with vertex set $\{m+1, \ldots, m+n\}$, so that the smallest vertex is $m+1$ and the number of vertices is n, but the set we have designated as K_n^m will better serve our purposes.

Notation 2.6 $LR(m)$ is the least n, if it exists, such that every 2-coloring of K_n^m has a large homogeneous set.

THIS IS THE LARGE RAMSEY THEOREM:

Theorem 2.7 *If COL is any 2-coloring of K_{N}, then, for every $m \geq 2$, there is a large homogeneous set whose smallest element is at least as large as m.*

Proof: Let COL be any 2-coloring of K_{N}. By Theorem 1.3, there exist an infinite set of vertices,

$$v_1 < v_2 < v_3 < \cdots,$$

and a color c such that, for all i, j, $COL(\{v_i, v_j\}) = c$. (This could be called an infinite homogeneous set.) Let i be such that $v_i \geq m$. The set

$$\{v_i, \ldots, v_{i+v_i-1}\}$$

is a homogeneous set that contains v_i elements and whose smallest element is v_i. Since $v_i \geq v_i$, it is a large set; hence it is a large homogeneous set. ∎

Theorem 2.8 *For every $m \geq 2$, $LR(m)$ exists.*

Proof:

Suppose, by way of contradiction, that there is some $m \geq 2$ such that $LR(m)$ does not exist. Then, for every $n \geq m-1$, there is some way to color K_n^m so that there is no large homogeneous set. Hence there exist the following:

1. COL_1, a 2-coloring of K_{m-1}^m that has no large homogeneous set
2. COL_2, a 2-coloring of K_m^m that has no large homogeneous set
3. COL_3, a 2-coloring of K_{m+1}^m that has no large homogeneous set

   $\vdots$

j. COL_j, a 2-coloring of K_{m+j-2}^m that has no large homogeneous set

   $\vdots$

We will use these 2-colorings to form a 2-coloring COL of K_{N} that has no large homogeneous set whose smallest element is at least as large as m.

Let $e_1, e_2, e_3, \ldots$ be a list of all unordered pairs of elements of N such that every unordered pair appears exactly once. We will color e_1, then e_2, etc.

How should we color e_1? We will color it the way an infinite number of the COL_i's color it. Call that color c_1. Then how to color e_2? Well, first consider ONLY the colorings that colored e_1 with color c_1. Color e_2 the way an infinite number of those colorings color it. And so forth.

We now proceed formally:

$$J_0 = \mathsf{N}$$

$$COL(e_1) = \begin{cases} \text{RED} & \text{if } |\{j \in J_0 \mid COL_j(e_1) = \text{RED}\}| \text{ is infinite} \\ \text{BLUE} & \text{otherwise} \end{cases}$$

$$J_1 = \{j \in J_0 \mid COL(e_1) = COL_j(e_1)\}$$

Let $i \geq 2$, and assume that $e_1, \ldots, e_{i-1}$ have been colored. Assume, furthermore, that J_{i-1} is infinite and, for every $j \in J_{i-1}$,

$$
\begin{aligned}
COL(e_1) = &\quad COL_j(e_1) \\
COL(e_2) = &\quad COL_j(e_2) \\
&\vdots \\
COL(e_{i-1}) = &\quad COL_j(e_{i-1})
\end{aligned}
$$

We now color e_i:

$$
COL(e_i) = \begin{cases} \text{RED} & \text{if } |\{j \in J_{i-1} \mid COL_j(e_i) = \text{RED}\}| \text{ is infinite} \\ \text{BLUE} & \text{otherwise} \end{cases}
$$

$$
J_i = \{j \in J_{i-1} \mid COL(e_i) = COL_j(e_i)\}
$$

One can show by induction that, for every i, J_i is infinite. Hence this process *never* stops.

Claim: If K_{N} is 2-colored with COL, then there is no large homogeneous set whose smallest element is at least as large as m.

Proof of Claim:

Suppose, by way of contradiction, that there is a large homogeneous set whose smallest element is at least as large as m. Without loss of generality, we can assume that the size of the large homogeneous set is equal to its smallest element. Let the vertices of that large homogeneous set be $v_1, v_2, \ldots v_{v_1}$, where $m \le v_1 < v_2 < \cdots < v_{v_1}$, and let the edges between those vertices be

$$
e_{i_1}, \ldots, e_{i_M},
$$

where $i_1 < i_2 < \cdots < i_M$ and $M = \binom{v_1}{2}$. For every $j \in J_{i_M}$, COL_j and COL agree on the colors of those edges. Choose $j \in J_{i_M}$ so that all the vertices of the large homogeneous set are elements of the vertex set of K^m_{m+j-2}. Then COL_j is a 2-coloring of the edges of K^m_{m+j-2} that has a large homogeneous set, in contradiction to the definition of COL_j.

End of Proof of Claim

Hence we have produced a 2-coloring of K_{N} that has no large homogeneous set whose smallest element is at least as large as m. This contradicts Theorem 2.7. Therefore, our initial supposition—that $LR(m)$ does not exist—is false. ▮

References

[1] J. Paris and L. Harrington. A mathematical incompleteness in Peano arithmetic. In J. Barwise, editor, *Handbook of Mathematical Logic*, pages 1133–1142. North-Holland, Amsterdam, 1977.

[2] F. Ramsey. On a problem of formal logic. *Proceedings of the London Mathematical Society*, 30:264–286, 1930. Series 2.