An amusing formula from *The TEXbook*

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The following formula appears in *The TEXbook* as a typesetting exercise (I have slightly modified it):

$$p_1(n) = \lim_{m\to\infty, m\in\mathbb{Z}} \sum_{k=0}^{\infty} \left(1 - \cos^{2m}\left(\frac{k!^n \pi}{n}\right)\right)$$

The expression has a clever mathematical meaning that is not discussed in the book.

In fact, the expression computes the largest prime that divides n, denoted $p_1(n)$. Here's how it works. First look at $k!^n/n$. If $k < p_1(n)$, then a factor of $p_1(n)$ appears in the denominator but not in the numerator, so $k!^n/n$ is not an integer. On the other hand, if $k \geq p_1(n)$, raising $k!$ to the nth power ensures that we have more than enough copies of each prime in the numerator to cancel the denominator, so $k!^n/n$ is an integer.

Working outward, when $k \geq p_1(n)$, the argument to the cosine is an integral multiple of π, so the cosine is 1 or -1. Raising it to the even power $2m$ yields 1, and subtraction from 1 leaves 0, so terms with $k \geq p_1(n)$ do not contribute to the summation. When $k < p_1(n)$, the angle is not an integral multiple of π, so its cosine is strictly between 1 and -1. Thus, the $2m$-th power approaches zero as m increases, so in the limit, these values of k contribute 1 each to the summation. The result is $\sum_{k=0}^{p_1(n)-1} 1 = p_1(n)$, as claimed.