Less Elegant Proofs that $\binom{2n}{n}$ and $C_n = \frac{1}{n+1}\binom{2n}{n}$ are Integers

By Bill Gasarch

1 Introduction

Notation 1.1 $\binom{n}{k}$ is $\frac{n!}{k!(n-k)!}$. C_n is $\frac{1}{n+1}\binom{2n}{n}$.

We all *know* that $\binom{2n}{n}$ is an integer since it solves a combinatorial problem. But if we didn't know that, it would not be obvious. We give several proofs that $\binom{2n}{n}$ is an integer, some of which do not require combinatorics.

We all *know* that C_n is an integer since it solves a combinatorial problem. But this is even less obvious than $\binom{2n}{n}$ being an integer! (That last punctuation is an explanation point, not a factorial.) We give several proofs that C_n is an integer, some of which do not require combinatorics.

2 Combinatorial Proof that $\binom{2n}{n}$ is an Integer

Consider the following problem: How many ways can you pick n distinct objects out of $2n$ distinct objects (without caring about the order). Whatever the answer is, it must be an integer. Elementary combinatorics shows that its $\binom{2n}{n}$.

I think its a great proof since we don't need to do any algebra. Others who don't know the combinatorics ahead of time might find it a very very odd proof. Such people also find the counting proof that $2^n = \sum_{i=0}^{n}\binom{n}{i}$ to be very very odd.

3 Proof by Combinatoric and Induction that $\binom{2n}{n}$ is an Integer

By a standard combinatorial argument one can show that

$$\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}.$$

Hence by induction $\binom{n}{k}$ is an integer.

This proof is a cheat. In order to do the combinatorial argument you need to already know that $\binom{n}{k}$ is the number of ways to choose k from n, and hence you already know that $\binom{2n}{n}$ is an integer

4 Proof by Algebra and Induction that $\binom{2n}{n}$ is an Integer

By an algebraic argument one can show that

$$\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}.$$

Hence by induction $\binom{n}{k}$ is an integer. Hence $\binom{2n}{n}$ is an integer.

This proof is not a cheat. It is completely algebraic. This proof also demonstrates the principle that it is sometimes easier to proof a harder theorem: We just wanted $\binom{2n}{n}$ to be an integer. However, this proof works well with $\binom{n}{k}$ and would be hard (impossible?) to push through a proof like this for just $\binom{2n}{n}$.

5 Proof by Number Theory that $\binom{2n}{n}$ is an Integer

Let p be a prime. We want to find x and y such that

1. x is the largest number such that p^x divides $(2n)!$

2. y is the largest number such that p^y divides $n!n!$

We then show that $x \geq y$ to complete the proof.

The number of factors of p in $m!$ is

$$\left\lfloor \frac{m}{p} \right\rfloor + \left\lfloor \frac{m}{p^2} \right\rfloor + \left\lfloor \frac{m}{p^3} \right\rfloor + \cdots$$

Hence

$$x = \left\lfloor \frac{2n}{p} \right\rfloor + \left\lfloor \frac{2n}{p^2} \right\rfloor + \left\lfloor \frac{2n}{p^3} \right\rfloor + \cdots$$

$$y = 2\left\lfloor \frac{n}{p} \right\rfloor + 2\left\lfloor \frac{n}{p^2} \right\rfloor + 2\left\lfloor \frac{n}{p^3} \right\rfloor + \cdots$$

To obtain $x \geq y$ it suffice to show that, for all real α, $\lfloor 2\alpha \rfloor \geq 2\lfloor \alpha \rfloor$. If α is an integer we get equality. If $n < \alpha < n+1$ then $2n < 2\alpha < 2n+2$ and hence $\lfloor \alpha \rfloor = n$ and $\lfloor 2\alpha \rfloor \geq 2n$. Hence we have $\lfloor 2\alpha \rfloor \geq 2\lfloor \alpha \rfloor$.

6 Incomplete Proof by Number Theory that $\binom{2n}{n}$ is an Integer

Is there a proof that $\binom{2n}{n}$ that just uses cancelling and not the kind of argument in Section 5? Here is an attempt at such.

$\binom{2n}{n}$ can be written as

$$\frac{2n}{n}\frac{2(n-1)}{n-1}\frac{2(n-2)}{n-2}\cdots\frac{2(n-(n-1)}{n-(n-1)} \times \frac{(2n-1)(2n-3)\cdots(2n-(2n-1))}{n!}.$$

This equals

$$2^n \times \frac{1 \times 3 \times 5 \times \cdots \times (2n-1)}{n!}$$

NEED TO FINISH. DO NOT WANT TO USE THAT 2^n DIVIDES $n!$ SINCE IF WE USE THAT TRICK WE MINE AS WELL HAVE USED THE PROOF IN THE LAST SECTION.

7 Combinatorial Proof that C_n is an Integer

This is well know but we include it for completness.

Let C_n be the number of ways to parenthesis $X \cdots X$ (n times). The parenthization should be such that if the operation is non-associative the answer is still unambigous.

Example 7.1

1. $n=0$: By convention we define $C_0=1$.

2. $n=1$: There is only one way to parenthsize X and that is by X.

3. $n=2$: There is only one way to parenthesis XX namely (XX) so $C_1=1$.

4. $n=3$: XXX can be parenthesized by either

   - $((XX)X)X$,
   - $(X(XX))X$,
   - $(XX)(XX)$,
   - $X((XX)X$,
   - $X(X(X(X)))$.

   Hence $C_3=5$.

We derive a recurrence for C_n. In any parentheization of X^n there will an Y, Z (nonempty) such that $X=(Y)(Z)$ where Y and Z are also parenthesized. Hence

$C_0=1$.

$$C_{n+1}=\sum_{i=0}^{n} C_i C_{n-i}$$

Let $C(x)=\sum_{i=0}^{\infty} C_n x^n$.

From the recurrence one can show that

$$C(x)=1+xC(x)^2.$$

Hence

$$C(x)=\frac{1-\sqrt{1-4x}}{2x}=\frac{2}{1+\sqrt{1-4x}}.$$

From this the Taylor series yields the answer.

If all you want to know is that C_n is an integer this actually is not an elegant proof.

8 Proof by Number Theory that C_n is an Integer

It is easy to show that $C_n = \binom{2n}{n} - \binom{2n}{n+1}$. By Section 5 we already have a proof that $\binom{2n}{n}$ is an integer that just uses number theory. It can easily be modified to show that $\binom{2n}{n+1}$ is an integer. Hence C_n is an integer.

Is there a proof that C_n is an integer that is similar to the proof in Section 5 that $\binom{2n}{n}$ is an integer?