Less Elegant Proofs that $\binom{2n}{n}$ and $C_n = \frac{1}{n+1} \binom{2n}{n}$ are Integers

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1 Introduction

Notation 1.1 $\binom{n}{k}$ is $\frac{n!}{k!(n-k)!}$. $C_n$ is $\frac{1}{n+1} \binom{2n}{n}$.

We all know that $\binom{2n}{n}$ is an integer since it solves a combinatorial problem. But if we didn’t know that, it would not be obvious. We give several proofs that $\binom{2n}{n}$ is an integer, some of which do not require combinatorics.

We all know that $C_n$ is an integer since it solves a combinatorial problem. But this is even less obvious than $\binom{2n}{n}$ being an integer! (That last punctuation is an explanation point, not a factorial.) We give several proofs that $C_n$ is an integer, some of which do not require combinatorics.

2 Combinatorial Proof that $\binom{2n}{n}$ is an Integer

Consider the following problem: How many ways can you pick $n$ distinct objects out of $2n$ distinct objects (without caring about the order). Whatever the answer is, it must be an integer. Elementary combinatorics shows that its $\binom{2n}{n}$.

I think its a great proof since we don’t need to do any algebra. Others who don’t know the combinatorics ahead of time might find it a very very odd proof. Such people also find the counting proof that $2^n = \sum_{i=0}^{n} \binom{n}{i}$ to be very very odd.

3 Proof by Combinatoric and Induction that $\binom{2n}{n}$ is an Integer

By a standard combinatorial argument one can show that

$$\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}.$$ 

Hence by induction $\binom{n}{k}$ is an integer.
This proof is a cheat. In order to do the combinatorial argument you need to already know that \( \binom{n}{k} \) is the number of ways to choose \( k \) from \( n \), and hence you already know that \( \binom{2n}{n} \) is an integer

4 Proof by Algebra and Induction that \( \binom{2n}{n} \) is an Integer

By an algebraic argument one can show that

\[
\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}.
\]

Hence by induction \( \binom{n}{k} \) is an integer. Hence \( \binom{2n}{n} \) is an integer.

This proof is not a cheat. It is completely algebraic. This proof also demonstrates the principle that it is sometimes easier to proof a harder theorem: We just wanted \( \binom{2n}{n} \) to be an integer. However, this proof works well with \( \binom{n}{k} \) and would be hard (impossible?) to push through a proof like this for just \( \binom{2n}{n} \).

5 Proof by Number Theory that \( \binom{2n}{n} \) is an Integer

Let \( p \) be a prime. We want to find \( x \) and \( y \) such that

1. \( x \) is the largest number such that \( p^x \) divides \( (2n)! \)
2. \( y \) is the largest number such that \( p^y \) divides \( n!n! \)

We then show that \( x \geq y \) to complete the proof.

The number of factors of \( p \) in \( m! \) is

\[
\left\lfloor \frac{m}{p} \right\rfloor + \left\lfloor \frac{m}{p^2} \right\rfloor + \left\lfloor \frac{m}{p^3} \right\rfloor + \cdots
\]

Hence

\[
x = \left\lfloor \frac{2n}{p} \right\rfloor + \left\lfloor \frac{2n}{p^2} \right\rfloor + \left\lfloor \frac{2n}{p^3} \right\rfloor + \cdots
\]
\[ y = 2 \left\lfloor \frac{n}{p} \right\rfloor + 2 \left\lfloor \frac{n}{p^2} \right\rfloor + 2 \left\lfloor \frac{n}{p^3} \right\rfloor + \cdots \]

To obtain \( x \geq y \) it suffice to show that, for all real \( \alpha \), \( \lfloor 2\alpha \rfloor \geq 2 \lfloor \alpha \rfloor \). If \( \alpha \) is an integer we get equality. If \( n < \alpha < n + 1 \) then \( 2n < 2\alpha < 2n + 2 \) and hence \( \lfloor \alpha \rfloor = n \) and \( \lfloor 2\alpha \rfloor \geq 2n \). Hence we have \( \lfloor 2\alpha \rfloor \geq 2 \lfloor \alpha \rfloor \).

6 Incomplete Proof by Number Theory that \( \binom{2n}{n} \) is an Integer

Is there a proof that \( \binom{2n}{n} \) that just uses cancelling and not the kind of argument in Section 5? Here is an attempt at such.

\( \binom{2n}{n} \) can be written as

\[
\frac{2n}{n} \frac{2(n - 1)}{n - 1} \frac{2(n - 2)}{n - 2} \cdots \frac{2(n - (n - 1))}{n - (n - 1)} \times \frac{(2n - 1)(2n - 3) \cdots (2n - (2n - 1))}{n!}
\]

This equals

\[
2^n \times \frac{1 \times 3 \times 5 \times \cdots \times (2n - 1)}{n!}
\]

NEED TO FINISH. DO NOT WANT TO USE THAT \( 2^n \) DIVIDES \( n! \) SINCE IF WE USE THAT TRICK WE MINE AS WELL HAVE USED THE PROOF IN THE LAST SECTION.

7 Combinatorial Proof that \( C_n \) is an Integer

This is well know but we include it for completness.

Let \( C_n \) be the number of ways to parenthesis \( X \cdots X \) (\( n \) times). The parenthization should be such that if the operation is non-associative the answer is still unambiguous.

Example 7.1
1. \( n = 0 \): By convention we define \( C_0 = 1 \).

2. \( n = 1 \): There is only one way to parenthesize \( X \) and that is by \( X \).

3. \( n = 2 \): There is only one way to parenthesize \( XX \) namely \( (XX) \) so \( C_1 = 1 \).

4. \( n = 3 \): \( XXX \) can be parenthesized by either
   - \( ((XX)X)X, \)
   - \( (X(XX))X, \)
   - \( (XX)(XX), \)
   - \( X((XX)X, \)
   - \( X(X(X(X))). \)

   Hence \( C_3 = 5 \).

   We derive a recurrence for \( C_n \). In any parentheization of \( X^n \) there will an \( Y, Z \) (nonempty) such that \( X = (Y)(Z) \) where \( Y \) and \( Z \) are also parenthesized. Hence \( C_0 = 1 \).

   \[
   C_{n+1} = \sum_{i=0}^{n} C_i C_{n-i}
   \]

   Let \( C(x) = \sum_{i=0}^{\infty} C_n x^n \).

   From the recurrence one can show that

   \[
   C(x) = 1 + xC(x)^2.
   \]

   Hence

   \[
   C(x) = \frac{1 - \sqrt{1 - 4x}}{2x} = \frac{2}{1 + \sqrt{1 - 4x}}.
   \]
From this the Taylor series yields the answer.

If all you want to know is that $C_n$ is an integer this actually is not an elegant proof.

8 Proof by Number Theory that $C_n$ is an Integer

It is easy to show that $C_n = \binom{2n}{n} - \binom{2n}{n+1}$. By Section 5 we already have a proof that $\binom{2n}{n}$ is an integer that just uses number theory. It can easily be modified to show that $\binom{2n}{n+1}$ is an integer. Hence $C_n$ is an integer.

Is there a proof that $C_n$ is an integer that is similar to the proof in Section 5 that $\binom{2n}{n}$ is an integer?