

Review of¹
The Pea and the Sun: A Mathematical Paradox
by Leonard Wapner
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Review by
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1 Introduction

Bill: The Banach-Tarski Theorem states that you can take a sphere of volume 1, break it into a finite number of pieces, then put them together again, and have a sphere of volume 2.

Grad Student: Math is broken!

High School Student: That's just wrong man!

Bill: Would it help if I told you that

- The proof uses the Axiom of Choice.
- The proof is non-constructive. Hence it will not give you a method for carrying out this construction.
- At least one of the pieces that you obtain is not measurable. This means that you cannot assign a volume to it in an intelligent fashion.

Grad Student and High School Student: No.

What Bill refers to as *The Banach-Tarski Theorem* is more often called *The Banach Tarski Paradox*. The title of the book comes from starting the procedure with a pea and doing it many times until you get a sphere the size of the sun. The book is about both the history leading up to the paradox, the paradox itself, and the reaction to it. The book does include enough math so that if you do not know the proof of the paradox ahead of time (I did not) then you can understand it. Note that you may understand the proof (I did) without liking the result (I didn't) or accepting it (I don't).

2 Summary of Contents

Chapter 1 is titled **History: A Cast of Characters**. This chapter has subchapters on Cantor, Banach, Tarski, Godel, and Cohen. There is some personal material but the book mostly concentrates on the history of the mathematics they did. Zermelo and Frankl also make an appearance, as does Hausdorff. This chapter also contains some easy math- mostly material on countability, unaccountability, and the axioms of set theory. The most interesting part of this chapter is the description of Hausdorff's theorem (later known as Hausdorff's paradox): The surface of a ball minus a relatively small number of points can be partitioned into three disjoint sets A, B, C such that $A, B, C, B \cup C$

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are all congruent. Results like this, and Vitali's theorem (there exists a non-measurable set) serve to remind us that results like the Banach-Tarski Paradox do not come all-of-a-sudden. They build on prior results.

Chapter 2 is titled **Jigsaw Fallacies and Other Curiosities**. This chapter talks about other odd things in mathematics, including Simpsons Paradox in statistics. These oddities are not directly related to the Banach-Tarski Paradox; however, they serve to remind us that mathematics has other odd things in it. Some can be resolved, but some not.

Chapter 3 is titled **Preliminaries**. This begins to get to the heart of the matter. They talk about set theory, isometries (mappings that preserve distance), scissor congruences (two polygons are congruent if you can take them apart with straight line cuts and rearrange one into the other). They prove that any two polygons of the same area are scissor-congruent. The same statement problem for polyhedra is false (This was Hilbert's third problem.) They also talk about equidecomposability which is a more careful way of looking at scissor congruences. The most fun part of this chapter is the lyrics to *Hotel Infinity* which is about Hilbert's Hotel. It is to the tune of *Hotel California*. (I hope someone releases it on You-Tube soon!)

Chapter 4 is titled **Baby BT's**. By this they mean mini-versions of the Banach-Tarski Paradox. This chapter has results that are startling, but not as startling as the full blown Banach-Tarski Paradox. One is Vitali's result that there are non-measurable sets. Another is the Sierpinski-Mazurkiewicz Paradox- there exists a set E of the plane that can be partitioned into $E_1 \cup E_2$ such that E, E_1, E_2 are all congruent. This proof does not use the Axiom of Choice.

Chapter 5 is titled **Statement and Proof of the Theorem**. I read Chapters 1,2,3,4 on a 4 hour train ride and understood most of it (I also knew about 1/3 of it ahead of time). Chapter 5 was harder but would have been much harder had I not read Chapters 1,2,3,4. Chapter 5 is understandable and does give the full correct proof. Understanding the proof of the Banach-Tarski Paradox does not make me like the result any better. A layperson may well understand most of Chapters 1,2,3,4 but not 5. That layperson will still have learned a great deal.

Chapter 6 is titled **Resolution**. No they do not resolve the paradox. They do suggest various ways people have understood the paradox. Some toss out the Axiom of Choice, some accept it at face value ("So math is weird. We knew that.") Some try to reinterpret. Nobody in this chapter claims that the result has any interpretation in the real real world.

Chapter 7 is titled **The Real World**. In this chapter there is an attempt to link the paradox to the real world. In particle physics there are cases where a collision creates new particles. Could this be BT in the real world? There are also attempts to link BT it to Chaos theory. These attempts seem to be done by serious people. Even so, I do not take them seriously.

Chapter 8 is titled **Yesterday, Today, and Tomorrow**. While it is a good read, it could be in a different book. It is about the state of math today and some speculation about its future.

3 Opinion

This was a great book for me. I never liked the Banach-Tarski Paradox but I had never really understood the proof; hence, I was not qualified to have a public opinion. Now I am. I still don't like it.

I recommend the book for anyone who is interested in the paradox and wants to know the history and context behind it. I would recommend reading the entire book (the first four chapters are a quick read) and not jumping to the proof.

I have one criticism which is more of a recommendation for the next version. The Axiom of Determinacy has been suggested as a replacement for the Axiom of Choice. One reason is that it implies that all sets of reals are measurable. This should be mentioned.