Proportional Division
Exposition by William Gasarch

1 Introduction

Whenever we say something like *Alice has a piece worth 1/2* we mean worth 1/2 TO HER.

Lets say we want Alice, Bob, Carol, to split a cake so that each one thinks they got $\geq 1/3$. This is called a *proportional division.* More generally, if $A_1, \ldots, A_n$ want to split a cake so that they all get $\geq 1/n$, that is called a *proportional division.*

For two people we can always use cut-and-choose which takes just one cut.

In all protocols we describe what the players DO and put in parenthesis advice as to what they SHOULD DO in their own best interest. If they have to choose a piece and the advice is to pick the bigger one, we omit the advice.

The players will be denoted Alice, Bob, Carol if there are 3 of them, and $A_1, \ldots, A_n$ if there are n of them. If we need to number pieces we will all them $P_1, P_2, \ldots$. There is no connection between person A_i and piece P_i.

For each protocol we will usually leave the following to the reader:

1. For each player A_i, if A_i follows the advice then, no matter what the other players do, A_i will get a proportional share.
2. For each player A_i, if A_i does not follow the advice then there is a scenario where they get less than a proportional share.

2 COME LATE Protocol

We present the 3-person COME LATE protocol and the n-person COME LATE protocol. It is called COME LATE since if (say) 8 people have already divided a cake so that each has 1/8 but they have not eaten it yet, then it someone else comes in (a ninth person) they can continue the protocol even though he was not there at the beginning.

Theorem 2.1 *There is a discrete protocol for 3 people to achieve a proportional division which uses 5 cuts.*

Proof:

Here is the protocol:

1. Alice and Bob do cut-and-choose (fairly). Alice has piece P_1, Bob has piece P_2. One cut.
2. Carol and Alice divide P_1 in ratio (1:2).
3. Carol and Alice divide P_1 in ratio (1:2).

Note that there are 5 cuts total.

We show that if Carol follows the advice then she gets $\geq 1/3$ independent of what anyone else does.

Say Carol values P_1 as v_1 and P_2 as v_2. All we know is that $v_1 + v_2 = 1$. If Carol cuts P_1 and P_2 into thirds equally then Carol gets

$$(1/3)v_1 + (1/3)v_2 = (1/3)(v_1 + v_2) = 1/3.$$

▮

Theorem 2.2 *For all $n \geq 2$ there is a discrete protocol for n people to achieve a proportional division which always uses approximately $n^2 \log n$.*

Proof:

For $n = 2$ people we use cut-and-choose. Note that this take one cut. Assume we already have a protocol for $n-1$ people.

1. $A_1, \ldots, A_{n-1}$ do the protocol for $n-1$ people. For $1 \leq i \leq n-1$ A_i has P_i and values it as $\geq \frac{1}{n-1}$.
2. For $1 \leq i \leq n-1$,
   (a) A_n cuts and A_i divide P_i in ratio $(1 : n-1)$.

We show that Carol gets $\geq 1/n$.

Say that, for all $1 \leq i \leq n-1$, Carol values P_i as v_i. All we know is that $v_1 + v_2 + \cdots + v_{n-1} = 1$. If Carol cuts each P_i into n equal pieces then Carol gets

$$(1/n)v_1 + (1/n)v_2 + \cdots + (1/n)v_{n-1} = (1/n)(v_1 + v_2 + \cdots + v_{n-1}) = 1/n.$$

How many cuts were used. Let $C(n)$ be the number of cuts for n people. $C(2) = 1$.

For n people the protocol first uses $C(n-1)$ cuts and then uses, for each $1 \le i \le n-1$, $n-1$ cuts. Hence

$$C(n) \le C(n-1) + (n-1)\log n.$$

One can show that $C(n) \le n^2 \log n + A$ for some constant A which we don't care about.

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Some notes.

1. No matter what the tastes are of $A_1, \ldots, A_n$ this protocol will take roughly $n^2 \log n)$ cuts. So this numbers is the best case as well as the worst case.

2. Is there a protocol that takes substantially less than $n^2 \log n$ cuts? YES- in the next section!

3 TRIM Protocol

We present the 3-person TRIM protocol and the n-person TRIM protocol. It is called TRIM since the players may do a lot of trimming of pieces of cake.

Theorem 3.1 *There is a discrete protocol for 3 people to achieve a proportional division which uses 3 cuts.*

Proof:

1. Alice cuts a piece (1/3). One cut.

2. Bob either trims the piece and puts the trim aside or not (If the piece is $> 1/3$ then trim down to $1/3$.) At most one cuts.

3. Carol takes the piece or not.

(a) If Carol takes the piece then Alice and Bob put the trim (if there is any) back on the cake and do cut-and choose with what is left. One cut.

(b) If Carol does not take the piece, and Bob trimmed it, then Bob gets the piece. Then Alice and Carol do cut-and-choose with whats left. One cut.

(c) If Carol does not take the piece, and Bob did not trim it, then Alice gets the piece. Then Bob and Carol do cut-and-choose. One cut.

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Theorem 3.2 *For all $n \geq 2$ there is a discrete protocol for n people to achieve a proportional division which uses at most $\frac{(n-1)n}{2}$ cuts.*

Proof:

For $n = 2$ people we use cut-and-choose. Note that this take one cut which is what the theorem says it should take. Assume we already have a protocol for $n - 1$ people.

1. A_1 cuts a piece $(1/n)$. One cut.

2. For $2 \leq i \leq n - 1$,

   (a) A_i either trims the piece and puts the trim aside or not (If the piece is $> 1/n$ then trim down to $1/n$.) At most one cuts for each i, so at most $n - 2$ cuts.

3. A_n takes the piece or not.

   (a) If A_n takes the piece then add the trim back to the rest of the cake and let $A_1, \ldots, A_{n-1}$ execute the $n - 1$ player protocol on all that A_n has not taken.

   (b) If someone trimmed the cake then let i be the largest index of such a player. A_i gets the piece. $A_1, \ldots, A_{i-1}, A_{i+1}, \ldots, A_n$ execute the $(n - 1)$-player protocol on whats left (including the trim that is added back).

How many cuts were used. Let x_n be the number of cuts for n people. $x_2 = 1$.

For n people the protocol first gives has one player cut a piece, and then offers $n-2$ players a chance to trim. Hence in the worse case this phase of the protocol uses $n-1$ cuts. Then the protocol executes the $n-1$-player protocol. Hence

$$x_n = n - 1 + x_{n-1} = x_{n-1} + n - 1.$$

Note the following:

$$\begin{aligned} x_2 = &\ 1 \\ x_3 = &\ x_2 + 2 = 1 + 2 \\ x_4 = &\ x_3 + 3 = 1 + 2 + 3 \end{aligned}$$

One can see (formally by induction, but that is not important for this course) that

$$x_n = 1 + 2 + \cdots + (n-1).$$

This is a known summation. It is $\frac{(n-1)n}{2}$.

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Some notes.

1. We will say that this protocol is $O(n^2)$. The formal definition is not important, suffice to say that the protocol takes roughly n^2.

2. Depending on the tastes of $A_1, \ldots, A_n$ this protocol may take far less than $O(n^2)$ cuts. In the best case this protocol would take $n-1$ cuts.

3. Is there a protocol that takes substantially less than $O(n^2)$ cuts? YES- in the next section!

4 DIVIDE & CONQUER (DC) Protocol

In this section we abbreviate DIVIDE & CONQUER by DC. We present the 4-person DC protocol and then the n-person DC protocol. It is called DIVIDE & CONQUER since the players split into two groups, one of which divides the Left side of the cake, and the other the right side of the cake. This type of protocol does not work with 3 people. If we (and we will) need to have some 3-player protocol we will use TRIM.

We denote a cake as a long line and think of it as a long rectangle. The following denotes that A, B, C made their cuts.

$- - - - A - - - - - - B - - - - - - C - - - - - - - - - - - - - - - -$

Theorem 4.1 *There is a discrete protocol for four people to achieve a proportional division which uses 5 cuts.*

Proof:

We call the players Alice, Bob, Carol, Donna.

1. Alice, Bob, and Carol all simultaneously cut the cake up and down (they each cut it in half). Three cuts. We assume just for notation that Alice's cut is the left most, then Bob's, then Carol's. Hence we have this:

   $- - - - A - - - - - - B - - - - - - C - - - - - - - - - - - - - - - -$

   Let L be the piece of cake to the LEFT of B. Let R be the piece of cake to the RIGHT of B. Not that (1) Alice and Bob both think that L is $\geq 1/2$. (2) Carol and Bob both think that R is $\geq 1/2$.

2. Donna picks either L or R.

   (a) If Donna picks L then (1) Alice and Donna do cut-and-choose on L, and (2) and Bob and Carol do cut-and-choose on R. Two cuts. KEY: Alice and Donna both think L is $\geq 1/2$, and Bob and Carol both think $R \geq 1/2$.

   (b) If Donna picks R then (1) Alice and Bob do cut-and-choose on L, and (2) and Carol and Donna do cut-and-choose on R. Two cuts. KEY: Alice and Bob both think L is $\geq 1/2$, and Carol and Donna both think $R \geq 1/2$.

Note that the protocol uses 5 cuts.

KEY: Bob was happy to *either* share L with one other person or R with one other people.

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Note 4.2 Compare this number-of-cuts with both the COME LATE and TRIM for $n = 4$.

Before we generalize this note that it was important that 4 was even. An odd number (3) of people initially cut the cake so there was a *middle cut* that we could use. What if there is an odd number of people total, so if all but one cut, there is an even number of people initially cutting the cake? KEY: The initially cutters won't be advised to cut evenly. We do the $n = 5$ case.

Theorem 4.3 *There is a discrete protocol for 5 people to achieve a proportional division which uses 5 cuts.*

Proof:

We call the players Alice, Bob, Carol, Donna, Edgar.

1. Alice, Bob, Carol, and Edgar all simultaneously cut the cake up and down (KEY: they each cut it (2/5,3/5).) Four cuts. We assume just for notation that the cuts are as follows:

   $- - - - A - - - - - - B - - - - - - C - - - - - - - - - - D - - - - - - -$

   Let L be the piece of cake to the LEFT of B. Let R be the piece of cake to the RIGHT of B. Not that (1) Alice, Bob think that L is $\geq 2/5$. (2) Bob, Carol, Edgar think that R is $\geq 3/5$.

2. Edgar picks either L or R. (Edgar picks L if he thinks $L \geq 2/5$ and R if he thinks $R \geq 3/5$. Note that he has to think one of these two.)

   (a) If Edgar picks L then (1) Alice and Edgar do cut-and-choose on L, and (2) Bob, Carol, Donna do a 3-person TRIM protocol on R. Four cuts (one for cut-and-choose, 3 for TRIM). KEY: Alice and Edgar think L is $\geq 2/5$, and Bob, Carol, Donna think $R \geq 3/5$.

   (b) If Donna picks R then (1) Alice and Bob do cut-and-choose on L, and (2) Carol, Donna, Edgar do 3-person TRIM protocol on R. Four cuts. KEY: Alice, Bob both think L is $\geq 2/5$, and Carol, Donna, Edgar both think $R \geq 3/5$.

Note that this takes 8 cuts.

KEY: Bob was happy to *either* share L with one other person or R with two other people.

▮

To generalize this protocol for n people we need two protocols: one for n even and one for n odd.

Theorem 4.4 *For all n there is a discrete protocol for n people to achieve a proportional division which uses $O(n \log n)$ cuts.*

Proof:

We call the players $A_1, \ldots, A_n$ Let the number of cuts be x_n.

For $n = 2$ we use Cut and Choose. For $n = 3$ we use TRIM. We describe two algorithms, one for n even and one for n odd. In each case we will call the protocol itself with a lower value of n. In the prior protocols we called the protocol itself for $n - 1$ players. Here we will call it with roughly $n/2$ players.

n even case. Note that $n - 1$ is odd

1. $A_1, \ldots, A_{n-1}$ all simultaneously cut the cake up and down (they each cut it in half). $n - 1$ cuts. We assume just for notation that this is the scenario:

   $--A_1---A_2-\ldots-A_{\frac{n}{2}-1}--A_{\frac{n}{2}}---A_{\frac{n}{2}+1}--\ldots--A_{n-1}-$

   Let L be the piece of cake to the LEFT of $A_{\frac{n}{2}}$. Let R be the piece of cake to the RIGHT of $A_{\frac{n}{2}}$. Note that (1) $A_1, \ldots, A_{\frac{n}{2}}$ think that L is $\geq 1/2$. (2) $A_{\frac{n}{2}}, , \ldots, A_{n-1}$ think that R is $\geq 1/2$.

2. A_n picks either L or R.

   (a) If A_n picks L then (1) $A_1, \ldots, A_{\frac{n}{2}-1}, A_n$ do the DC protocol with $n/2$ players on L, (2) $A_{\frac{n}{2}}, A_{\frac{n}{2}+1}, \ldots, A_{n-1}$ do the DC protocol with $n/2$ players on R. $2x_{n/2}$ cuts. KEY: (1) $A_1, \ldots, A_{\frac{n}{2}-1}, A_n$ all think $L \geq 1/2$. (2) $A_{\frac{n}{2}}, A_{\frac{n}{2}+1}, \ldots, A_{n-1}$ all think $R \geq 1/2$.

   (b) If A_n picks R then (1) $A_1, \ldots, A_{\frac{n}{2}}$ do the DC protocol with $n/2$ players on L, (2) $A_{\frac{n}{2}+1}, A_{\frac{n}{2}+1}, \ldots, A_{n-1}, A_n$ do the DC protocol with $n/2$ players on R. $2x_{n/2}$ cuts. KEY: (1) $A_1, \ldots, A_{\frac{n}{2}}$, all think $L \geq 1/2$. (2) $A_{\frac{n}{2}+1}, \ldots, A_n$ all think $R \geq 1/2$.

Note that the protocol uses $n - 1 + 2x_{n/2}$ cuts.

KEY: $A_{\frac{n}{2}}$ is happy to *either* share L with $n/2 - 1$ other person or share R with $n/2 - 1$ other people.

n odd protocol.

1. $A_1, \ldots, A_{n-1}$ all simultaneously cut the cake up and down (KEY: They each cut it $(\frac{n-1}{2n}, \frac{n+1}{2n})$.) We assume just for notation that this is the scenario:

   $-A_1 - -A_2 - \ldots - A_{\frac{n-1}{2}-1} - - - A_{\frac{n-1}{2}} - - - A_{\frac{n-1}{2}+1} - \ldots - A_{n-1} - -$

   Let L be the piece of cake to the LEFT of $A_{\frac{n-1}{2}}$. Let R be the piece of cake to the RIGHT of $A_{\frac{n-1}{2}}$. Note that (1) $A_1, \ldots, A_{\frac{n-1}{2}}$ think that L is $\ge \frac{n-1}{2n}$. $n-1$ cuts. (2) $A_{\frac{n-1}{2}}, , \ldots, A_{n-1}$ think that R is $\ge \frac{n+1}{2n}$.

2. A_n picks either L or R. (He picks L if he thinks $L \ge \frac{n-1}{2n}$ and R if he thinks $R \ge \frac{n+1}{2n}$. He must think one of these.)

   (a) If A_n picks L then (1) $A_1, \ldots, A_{\frac{n-1}{2}-1}, A_n$ do the DC protocol with $(n-1)/2$ players on L, (2) $A_{\frac{n-1}{2}}, A_{\frac{n-1}{2}+1}, \ldots, A_{n-1}$ do the DC protocol with $(n+1)/2$ players on R. $x_{(n-1)/2} + x_{(n+1)/2}$ cuts. KEY: (1) $A_1, \ldots, A_{\frac{n-1}{2}-1}, A_n$ all think $L \ge (n-1)/2n$. (2) $A_{\frac{n-1}{2}}, A_{\frac{n-1}{2}+1}, \ldots, A_{n-1}$ all think $R \ge (n+1)/2n$. $x_{(n-1)/2} + x_{(n+1)/2}$ cuts.

   (b) If A_n picks R then (1) $A_1, \ldots, A_{\frac{n-1}{2}}$ do the DC protocol with $(n-1)/2$ players on L, (2) $A_{\frac{n-1}{2}+1}, A_{\frac{n-1}{2}+1}, \ldots, A_{n-1}, A_n$ do the DC protocol with $(n+1)/2$ players on R. $x_{(n-1)/2} + x_{(n+1)/2}$ cuts. KEY: (1) $A_1, \ldots, A_{\frac{n-1}{2}}$, all think $L \ge (n-1)/2n$. (2) $A_{\frac{n-1}{2}+1}, \ldots, A_n$ all think $R \ge (n+1)/2n$. $x_{(n-1)/2} + x_{(n+1)/2}$ cuts.

The KEY to the protocol is that $A_{\frac{n}{2}}$ is happy splitting either L with $(n-1)/2n$ people or R with $(n+1)/2n$ people.

How does x_n grow? To recap we have

$x_2 = 1$

$x_3 = 3$

$x_n = n - 1 + 2x_{n/2}$ if n is even

$x_n = n - 1 + x_{(n-1)/2} + x_{(n+1)/2}$ if n is odd.

Lets first look at what happens when n is a power of 2 to get an idea of how fast x_n grows. Let $n = 2^k$.

$x_{2^k} = 2^k + 2x_{2^{k-1}}$

$x_{2^k} = 2^k + 2(2^{k-1} + 2x_{2^{k-2}}) = 2^k + 2^k + 2^2 x_{2^{k-2}} = 2 \times 2^k + 2^2 x_{2^{k-2}}$.

We can keep doing this:

$$
\begin{aligned}
x_{2^k} &= 2 \times 2^k + 2^2 x_{2^{k-2}} \\
&= 2 \times 2^k + 2^2(2^{k-2} + 2x_{2^{k-3}}) \\
&= 2 \times 2^k + 2^k + 2^3 x_{2^{k-3}} \\
&= 3 \times 2^k + 2^3 x_{2^{k-3}}
\end{aligned}
$$

One can see (formally by induction, but that is not important for this course) that, for all $1 \leq i \leq k-1$

$$x_{2^k} = i \times 2^k + 2^i x_{2^{k-i}}.$$

Plugging in $i = k - 1$ we obtain

$$x_{2^k} = (k-1) \times 2^k + 2^{k-1} x_2 = (k-1)2^k + 2^{k-1}.$$

We want to express this in terms of n using $n = 2^k$ and $k = \lg n$.

$$x_n = (\lg n - 1)n + n/2.$$

This is bounded above by $O(n \log n)$.

The general case where n is not a power of two is similar but messy. We can bound it above by $O(n \log n)$.

Hence the protocol takes at most $O(n \log n)$ cuts. ∎

Some notes:

1. The number of cuts does not depend on peoples tastes. $O(n \log n)$ is both an upper and lower bound.

2. Can we do better? NO- it is KNOWN that using algorithms OF THIS TYPE it requires roughly $n \log n$ cuts.