Equations and Colorings: Rado's Theorem
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1 Introduction

Everything in this paper was proven by Rado [2] (but see also [1]).

Do you think the following is TRUE or FALSE?

For any 17-coloring $COL : \mathsf{N} \rightarrow [17]$ *there exists* e_1, e_2, e_3 *such that*

$$COL(e_1) = COL(e_2) = COL(e_3)$$

and

$$2e_1 + 5e_2 - e_3 = 0.$$

2 FALSE

The statement is FALSE. Our first attempt at finding a 17-coloring will not quite work, but our second one will.

First Attempt

$$COL(n) \text{ is the number between 0 and 16 that is } \equiv n \pmod{17}.$$

Assume $COL(e_1) = COL(e_2) = COL(e_3)$. We will try to show that

$$2e_1 + 5e_2 - e_3 \neq 0.$$

Assume, by way of contradiction, that

$$2e_1 + 5e_2 - e_3 = 0.$$

Let e be such that $e_1 \equiv e_2 \equiv e_3 \equiv e \pmod{17}$ and $0 \leq e \leq 16$. Then

$$0 = 2e_1 + 5e_2 - e_3 \equiv 2e + 5e - e \equiv 6e \pmod{17}.$$

Hence $6e \equiv 0 \pmod{17}$. Since 6 has an inverse mod 17, we obtain $e \equiv 0 \pmod{17}$. We have *not* arrived at a contradiction. We have just established that if

$$COL(e_1) = COL(e_2) = COL(e_3)$$

and

$$2e_1 + 5e_2 - e_3 = 0.$$

Then $COL(e_1) = COL(e_2) = COL(e_3) = 0$.

Hence we will do a similar coloring but do something else when $n \equiv 0 \pmod{17}$.

Second Attempt

Given n let i, n' be such that 17^i divides n, 17^{i+1} does not divide n, and $n = 17^i n'$.

We define the coloring as follows:

$$COL(n) \text{ is the number between 1 and 16 that is } \equiv n' \pmod{17}.$$

NOTE- $COL(n)$ will never be 0. Hence this is really a 16-coloring.
Assume

$$COL(e_1) = COL(e_2) = COL(e_3).$$

We show that

$$2e_1 + 5e_2 - e_3 \neq 0.$$

Let $i, j, k, e_1', e_2', e_3', e$ be such that

1. 17^i divides e_1, 17^{i+1} does not divide e_1, $e_1 = 17^i e_1'$.
2. 17^j divides e_2, 17^{j+1} does not divide e_2, $e_2 = 17^j e_2'$.
3. 17^k divides e_3, 17^{k+1} does not divide e_3, $e_3 = 17^j e_3'$.
4. $e_1' \equiv e_2' \equiv e_3' \equiv e \pmod{17}$

If

$$2e_1 + 5e_2 - e_3 = 0$$

then

$$2 \times 17^i e_1' + 5 \times 17^j e_2' - 17^k e_3' = 0.$$

Every mathematical bone in my body wants to cancel some of the 17's. There are cases. All $\equiv$ are mod 17.

1. $i < j \le k$ or $i < k \le j$.

   $$2 \times 17^i e_1' + 5 \times 17^j e_2' - 17^k e_3' = 0.$$

   Divide by 17^i.

   $$2 \times e_1' + 5 \times 17^{j-i} e_2' - 17^{k-i} e_3' = 0.$$

   We take this equation mod 17.

   $$2e_1' \equiv 2e \equiv 0.$$

   Since 2 has an inverse mod 17 we have $e = 0$. This contradicts that $e \neq 0$.

2. $i = j < k$.

$$2 \times 17^i e_1' + 5 \times 17^i e_2' - 17^k e_3' = 0.$$

Divide by 17^i.

$$2 \times e_1' + 5 \times 17^{j-i} e_2' - 17^{k-i} e_3' = 0.$$

We take this equation mod 17.

$$2e_1' + 5e_2' \equiv 7e \equiv 0.$$

Since 7 has an inverse mod 17 we have $e = 0$. This contradicts that $e \neq 0$.

3. Rather than go through all of the cases in detail, we say what results in all caes, including those above.

   (a) $i < j \le k$ or $i < k \le j$: $2e \equiv 0$.
   (b) $i = j < k$: $2e + 5e \equiv 0$.
   (c) $i = k < j$: $2e - e \equiv 0$.
   (d) $i = j = k$: $2e + 5e - 3e \equiv 0$.
   (e) $j < i \le k$ or $j < k \le i$: $5e \equiv 0$.
   (f) $j = k < i$: $2e - e \equiv 0$.
   (g) $k < i = j$: $-e \equiv 0$.

There were 7 cases. Each corresponded to a combination of the coefficients. The key is that every combination was relatively prime to 17. The reader should be able to prove the following.

Theorem 2.1 *Let $b_1, \ldots, b_n \in \mathsf{Z}$. If there exists c that is relatively prime to every nonempty subsum of $\{b_1, \ldots, b_n\}$ then there is a $c-1$-coloring of N such that there is no $e_1, \ldots, e_n \in \mathsf{N}$ with*

$$COL(e_1) = \cdots = COL(e_n)$$

and

$$b_1 e_1 + \cdots + b_n e_n = 0.$$

3 TRUE

So is there any $b_1, \ldots, b_n$ so that a positive statement about colorings is true. For what $b_1, \ldots, b_n$ could the premise of Theorem 2.1 be false? The only way is if some nontempty subset of $\{b_1, \ldots, b_n\}$ sums to 0.

Theorem 3.1 *Let $b_1, \ldots, b_n \in \mathsf{Z}$. Assume there exists a nonempty subset of $\{b_1, \ldots, b_n\}$ that sums to 0. For all c, for all c-coloring of N there exists $e_1, \ldots, e_n \in \mathsf{N}$ with*

$$COL(e_1) = \cdots = COL(e_n)$$

and

$$b_1e_1 + \cdots + b_ne_n = 0.$$

Before proving this theorem we talk about how to go about it. Lets use

$$5e_1 + 6e_2 - 11e_3 + 7e_4 - 2e_5 = 0$$

as an example. Note that the first three coefficients add to 0: $5 + 6 - 11 = 0$. We are thinking about colorings. OH, we can use van der Waerden's theorem!

Theorem 3.2 (van der Waerden) *For all k, for all c, for all c-colorings $COL : \mathsf{N} \rightarrow [c]$ there exists a, d such that*

$$COL(a) = COL(a + d) = COL(a + 2d) = \cdots COL(a + (k-1)d).$$

We will actually use the following easy corollary

Theorem 3.3 (van der Waerden) *For all $x_1, \ldots, x_k \in \mathsf{Z}$, for all c, for all c-colorings $COL : \mathsf{N} \rightarrow [c]$ there exists a, d such that*

$$COL(a) = COL(a + x_1d) = COL(a + x_2d) = \cdots COL(a + x_kd).$$

We use the $k = 5$ case. Is there a choice of x_1, x_2, x_3, x_4, x_5 that will give us our theorem?
Say that $e_i = a + x_id$. Then

$$\begin{aligned}
5e_1 + 6e_2 - 11e_3 + 7e_4 - 2e_5 = \; & 5(a + x_1d) + 6(a + x_2d) - 11(a + x_3d) + 7(a + x_4d) - 2(a + x_5d) \\
= \; & (5 + 6 - 11)a + d(5x_1 + 6x_2 - 11x_3) + (7 - 2)a + d(7x_4 - 2x_5). \\
= \; & (5 + 6 - 11)a + d(5x_1 + 6x_2 - 11x_3 + 7x_4 - 2x_5) + 5a.
\end{aligned}$$

GOOD NEWS: The first a has coefficeint (5+6-11)=0.
GOOD NEWS: We can pick x_1, x_2, x_3, x_4, x_5 to make the $5x_1+6x_2-11x_3+7x_4-2x_5 = 0$.
BAD NEWS: The $5a$ looks hard to get rid of.
It would be really great if we did not have that '$5a$' term.
Hence we need a variant of van der Waerden's theorem.
The following is true and will be proved in the Section 4

Lemma 3.4 *For all k, s, c, for any c-coloring COL of* N, *there exists a, d such that*

$$COL(a) = COL(a + d) = \cdots = COL(a + (k - 1)d) = COL(sd).$$

We now state an easy corollary of this. We still call it a lemma since we don't really care about it for itself, only for what it can do for us.

Lemma 3.5 *For all $x_1, \ldots, x_m \in \mathsf{Z}$, for all $s \in \mathsf{N}$, for all $c \in \mathsf{N}$, for any c-coloring COL of N there exists a, d such that*

$$COL(a) = COL(a + x_1 d) = COL(a + x_2 d) = \cdots = COL(a + x_m d) = COL(sd).$$

We now restate and prove the main theorem of this section.

Theorem 3.6 *Let $b_1, \ldots, b_n \in \mathsf{Z}$. Assume there exists a nonempty subset of $\{b_1, \ldots, b_n\}$ that sums to 0. For all c, for all c-coloring of N, there exists $e_1, \ldots, e_n \in \mathsf{N}$ with*

$$COL(e_1) = \cdots = COL(e_n)$$

and

$$b_1 e_1 + \cdots + b_n e_n = 0.$$

Proof: The cases of $n = 1$ and $n = 2$ are easy and left to the reader. Hence we assume $n \geq 3$. If any of the b_i's are 0 then we can omit the term with that b_i. So we can assume that $(\forall i)[b_i \neq 0]$.

By renumbering we can assume that there is an $m \leq n$ such that

$$\sum_{i=1}^{m} b_i = 0.$$

Let COL be a c-coloring of N. We will determine $x_1, \ldots, x_m \in \mathsf{Z} - \{0\}$ and $s \in \mathsf{N}$ later. By Lemma 3.5 there exists a, d such that

$$COL(a) = COL(a + x_1 d) = COL(a + x_2 d) = \cdots = COL(x + x_m d) = COL(sd).$$

We will let

$$e_1 = a + x_1 d,$$

$$e_2 = a + x_2 d,$$

$$\vdots$$

$$e_m = a + x_m d,$$

and

$$e_{m+1} = \cdots = e_n = sd$$

Then

$$\sum_{i=1}^{n} b_i e_i = \sum_{i=1}^{m} b_i e_i + \sum_{i=m+1}^{n} b_i e_i = \sum_{i=1}^{m} b_i(a + x_i d) + \sum_{i=m+1}^{n} b_i sd$$

This is equal to

$$a \sum_{i=1}^{m} b_i + d \sum_{i=1}^{L} b_i x_i + sd \sum_{i=m+1}^{n} b_i$$

KEY: $\sum_{i=1}^{m} b_i = 0$ so the first term drops out.

KEY: All of the remaining terms have a factor of d. If we want to set this to 0 we can cancel the d's. Hence we need $x_1, \ldots, x_n \in \mathsf{Z} - \{0\}$ and $s \in \mathsf{N}$ such that the following happens.

$$\sum_{i=1}^{m} b_i x_i + s \sum_{i=m+1}^{n} b_i = 0$$

Let $\sum_{i=m+1}^{n} b_i = B$. Then we rewrite this as

$$\sum_{i=1}^{m} b_i x_i + sB = 0$$

We can take

$$s = |mb_1 \cdots b_m|$$

$$x_1 = -\frac{sB}{b_1}$$

$$x_2 = -\frac{sB}{b_2}$$

$$\vdots$$

$$x_m = -\frac{sB}{b_m}.$$

∎

4 Proof of that VDW-type Theorem

We prove a theorem that looks stronger than Lemma 3.4 but is actually equivalent (by a compactness argument).

Lemma 4.1 *For all k, s, c, there exists $U = U(k,s,c)$ such that for every c-coloring $COL : [U] \rightarrow [c]$ there exists a, d such that*

$$COL(a) = COL(a+d) = \cdots = COL(a+(k-1)d) = COL(sd).$$

Proof: We prove this by induction on c. Clearly, for all k, s,

$$U(k,s,1) = \max\{k,s\}.$$

We assume $U(k,s,c-1)$ exists and show that $U(k,s,c)$ exists. We will show that

$$U(k,s,c) \le W((k-1)sU(k,s,c-1)+1,c).$$

Let COL be a coloring of $[W((k-1)sU(k,s,c-1)+1,c)]$. By the definition of W there exists a, d such that

$$COL(a) = COL(a+d) = \cdots = COL(a+(k-1)sU(k,s,c-1)).$$

Assume the color is RED.
1) $a, a+d, \ldots, a+(k-1)d$ are all RED. If sd is also RED then we are done. So we assume sd is NOT RED.
2) $a, a+2d, a+4d, \ldots, a+2(k-1)d$ are all RED. If $2sd$ is also RED then we are done. So we assume $2sd$ is NOT RED.
$\vdots$
$U(k,s,c-1)$) a,$a+U(k,s,c-1)d$,$a+2U(k,s,c-1)d$,…, $a+(k-1)U(k,s,c-1)d$ are all RED. If $U(k,s,c-1)sd$ is RED then we are done. So we assume $U(k,s,c-1)sd$ is NOT RED.

By the above we know that $sd, 2sd, 3sd, \ldots, U(k,s,c-1)sd$ are all NOT RED.
Consider the coloring $COL' : [U(k,s,c-1)] \rightarrow [c-1]$ defined by

$$COL'(x) = COL(xsd).$$

The KEY is that NONE of these will be colored RED so there are only $c-1$ colors.
By the inductive hypothesis there exists a', d' such that

$$COL'(a') = COL'(a'+d') = \cdots = COL'(a'+(k-1)d') = COL'(sd')$$

so

$$COL(a'sd) = COL(a'sd+d'sd) = \cdots = COL(a'sd+(k-1)d'sd) = COL(sd'sd)$$

Let $A = a'sd$ and $D = d'sd$. Then

$$COL(A) = COL(A+D) = \cdots = COL(A+(k-1)D = COL(sD).$$

▮

5 The Abridged Rado's Theorem

By Combining Theorem 2.1 and 3.1 we obtain what [1] refers to as *The Abridged Rado's Theorem.* In this section we state both the Abridged Rado's theorem and the full Rado Theorem.

Definition 5.1 A set of integers $(b_1, \ldots, b_n)$ is *regular* if the following holds: *For all c, for all c-colorings* $COL\mathsf{N} \rightarrow [c]$ *there exists* $e_1, \ldots, e_n$ *such that*

$$COL(e_1) = \cdots = COL(e_n),$$

$$\sum_{i=1}^{n} b_i e_i = 0.$$

The Abridged Rado's Theorem:

Theorem 5.2 $(b_1, \ldots, b_n)$ *is regular iff there exists some nonempty subset of* $\{b_1, \ldots, b_n\}$ *that sums to 0.*

6 The Full Rado's Theorem

The full Rado's Theorem is about systems of equations. We first view VDW's theorem as a system of equations. Lets take VDW's theorem with $k = 4$. It is usually written as

For all c, for all c-colorings $COL : N \rightarrow [c]$*, there exists* a, d *such that*

$$COL(a) = COL(a+d) = COL(a+2d) = COL(a+4d),$$

We rewrite this in terms of equationS.

For all c, for all c-colorings $COL : N \rightarrow [c]$*, there exists* e_1, e_2, e_3, e_4 *such that*

$$COL(a) = COL(a+d) = COL(a+2d) = COL(a+4d),$$

$$\begin{aligned} e_2 - e_1 &= e_3 - e_2 \\ e_2 - e_1 &= e_4 - e_3 \end{aligned}$$

We rewrite these equations:

$$\begin{aligned} 0e_4 - e_3 + 2e_2 - e_1 &= 0 \\ -e_4 + e_3 + e_2 - e_1 &= 0 \end{aligned}$$

Let A be the matrix:

$$\begin{pmatrix} 0 & -1 & 2 & -1 \\ -1 & 1 & 1 & -1 \end{pmatrix}$$

VDW for $k = 4$ can be rewritten as
For all c, for all c-colorings $COL\mathsf{N} \rightarrow [c]$ *there exists* $\vec{e} = e_1, \ldots, e_n$ *such that*

$$COL(e_1) = \cdots = COL(e_n),$$

$$A\vec{e} = \vec{0}.$$

What other matrices have this property?

Definition 6.1 A matrix A of integers is *regular* if if the following holds: *For all c, for all c-colorings* $COL\mathsf{N} \rightarrow [c]$ *there exists* $\vec{e} = e_1, \ldots, e_n$ *such that*

$$COL(e_1) = \cdots = COL(e_n),$$

$$A\vec{e} = \vec{0}.$$

Definition 6.2 A matrix A satisfies the *columns condition* if the columns can be ordered $\vec{c}_1, \ldots, \vec{c}_n$ and the set $\{1, \ldots, n\}$ can be partitioned into nonempty contigous sets $I_1, \ldots, I_k$ such that

$$\sum_{i \in I_1} \vec{c}_i = \vec{0},$$

For all j, $2 \leq j \leq k$, $\sum_{i \in I_j} \vec{c}_i$ can be written as a linear combination of the vectors $\{c_i\}_{i \in I_1 \cup \cdots \cup I_{j-1}}$.

The Full Rado's Theorem:

Theorem 6.3 *A is regular iff A satisfies the columns condition.*

PROOF- WILL FILL IN LATER

References

[1] R. Graham, A. Rothchild, and J. Spencer. *Ramsey Theory.* Wiley, 1990.

[2] R. Rado. Studien zur kombinatorik. *Mathematische Zeitschrift*, pages 424–480, 1933.