Equations and Infinite Colorings
Exposition by Stephen Fenner and William Gasarch

1 Introduction

Do you think the following is TRUE or FALSE?

For any $\aleph_0$-coloring of the reals, $COL : \mathsf{R} \rightarrow \mathsf{N}$ there exist distinct e_1, e_2, e_3, e_4 such that

$$COL(e_1) = COL(e_2) = COL(e_3) = COL(e_4),$$

$$e_1 + e_2 = e_3 + e_4.$$

It turns out that this question is equivalent to the negation of CH. Komjáth (3) claims that Erdős proved this result. The prove we give is due to Davies (1).

Definition 1.1 The *Continuum Hypothesis* (CH) is the statement that there is no order of infinity between that of N and R. It is known to be independent of Zermelo-Frankel Set Theory with Choice (ZFC).

Definition 1.2 ω_1 is the first uncountable ordinal. ω_2 is the second uncountable ordinal.

Fact 1.3

1. *If CH is true, then there is a bijection between R and ω_1. This has the counter-intuitive consequence: there is a way to list the reals:*

$$x_0, x_1, x_2, \ldots, x_\alpha, \ldots$$

   as $\alpha \in \omega_1$ such that, for all $\alpha \in \omega_1$, the set $\{x_\beta \mid \beta < \alpha\}$ is countable.

2. *If CH is false, then there is an injection from ω_2 to R. This has the consequence that there is a list of distinct reals:*

$$x_0, x_1, x_2, \ldots, x_\alpha, \ldots, x_{\omega_1}, x_{\omega_1+1}, \ldots, x_\beta, \ldots$$

   where $\alpha \in \omega_1$ and $\beta \in [\omega_1, \omega_2)$.

2 CH $\Rightarrow$ FALSE

Definition 2.1 Let $X \subseteq \mathsf{R}$. Then $CL(X)$ is the smallest set $Y \supseteq X$ of reals such that

$$a, b, c \in Y \;\Rightarrow\; a + b - c \in Y.$$

Lemma 2.2

1. *If X is countable then $CL(X)$ is countable.*

2. *If $X_1 \subseteq X_2$ then $CL(X_1) \subseteq CL(X_2)$.*

Proof:
1) Assume X is countable. $CL(X)$ can be defined with an ω-induction (that is, an induction just through ω).

$$\begin{aligned} C_0 &= X \\ C_{n+1} &= C_n \cup \{a+b-c \mid a,b,c \in C_n\} \end{aligned}$$

One can easily show that $CL(X) = \cup_{i=0}^{\infty} C_i$ and that this set is countable.
2) This is an easy exercise. ∎

Theorem 2.3 *Assume CH is true. There exists an $\aleph_0$-coloring of R such that there are no distinct e_1, e_2, e_3, e_4 such that*

$$COL(e_1) = COL(e_2) = COL(e_3) = COL(e_4),$$

$$e_1 + e_2 = e_3 + e_4.$$

Proof: Since we are assuming CH is true, we have, by Fact 1.3.1, there is a bijection between R and ω_1. If $\alpha \in \omega_1$ then x_α is the real associated to it. We can picture the reals as being listed out via

$$x_0, x_1, x_2, x_3, \ldots, x_\alpha, \ldots$$

where $\alpha < \omega_1$.

Note that every number has only countably many numbers less than it in this ordering.

For $\alpha < \omega_1$ let

$$X_\alpha = \{x_\beta \mid \beta < \alpha\}.$$

Note the following:

1. For all α, X_α is countable.
2. $X_0 \subset X_1 \subset X_2 \subset X_3 \subset \cdots \subset X_\alpha \subset \cdots$
3. $\bigcup_{\alpha<\omega_1} X_\alpha = \mathsf{R}$.

We define another increasing sequence of sets Y_α by letting

$$Y_\alpha = CL(X_\alpha).$$

Note the following:

1. For all α, Y_α is countable. This is from Lemma 2.2.1.
2. $Y_0 \subset Y_1 \subset Y_2 \subset Y_3 \subset \cdots \subset Y_\alpha \subset \cdots$. This is from Lemma 2.2.2.
3. $\bigcup_{\alpha<\omega_1} Y_\alpha = \mathsf{R}$.

We now define our last sequence of sets:

For all $\alpha < \omega_1$,

$$Z_\alpha = Y_\alpha - \left(\bigcup_{\beta<\alpha} Y_\beta \right).$$

Note the following:

1. Each Z_α is finite or countable.
2. The Z_α form a partition of R.

We will now define an $\aleph_0$-coloring of R. For each Z_α, which is countable, assign colors from ω to Z_α's elements in some way so that no two elements of Z_α have the same color.

Assume, by way of contradiction, that there are distinct e_1, e_2, e_3, e_4 such that

$$COL(e_1) = COL(e_2) = COL(e_3) = COL(e_4)$$

and

$$e_1 + e_2 = e_3 + e_4.$$

Let $\alpha_1, \alpha_2, \alpha_3, \alpha_4$ be such that $e_i \in Z_{\alpha_i}$. Since all of the elements in any Z_α are colored differently, all of the α_i's are different. We will assume $\alpha_1 < \alpha_2 < \alpha_3 < \alpha_4$. The other cases are similar. Note that

$$e_4 = e_1 + e_2 - e_3.$$

and

$$e_1, e_2, e_3 \in Z_{\alpha_1} \cup Z_{\alpha_2} \cup Z_{\alpha_3} \subseteq Y_{\alpha_1} \cup Y_{\alpha_2} \cup Y_{\alpha_3} = Y_{\alpha_3}.$$

Since $Y_{\alpha_3} = CL(X_{\alpha_3})$ and $e_1, e_2, e_3 \in Y_{\alpha_3}$, we have $e_4 \in Y_{\alpha_3}$. Hence $e_4 \notin Z_{\alpha_4}$. This is a contradiction. ∎

What was it about the equation

$$e_1 + e_2 = e_3 + e_4$$

that made the proof of Theorem 2.3 work? Absolutely nothing:

Theorem 2.4 *Let $n \geq 2$. Let $a_1, \ldots, a_n \in$ R be nonzero. Assume CH is true. There exists an $\aleph_0$-coloring of* R *such that there are no distinct $e_1, \ldots, e_n$ such that*

$$COL(e_1) = \cdots = COL(e_n),$$

$$\sum_{i=1}^{n} a_i e_i = 0.$$

Proof sketch: Since this prove is similar to the last one we just sketch it.

Definition 2.5 Let $X \subseteq R$. $CL(X)$ is the smallest superset of X such that the following holds:

For all $m \in \{1, \ldots, n\}$ and for all $e_1, \ldots, e_{m-1}, e_{m+1}, \ldots, e_n$,

$$e_1, \ldots, e_{m-1}, e_{m+1}, \ldots, e_n \in CL(X) \Rightarrow -(1/a_m) \sum_{i \in \{1,\ldots,n\}-\{m\}} a_i e_i \in CL(X).$$

Let X_α, Y_α, Z_α be defined as in Theorem 2.3 using this new defintion of CL. Let COL be defined as in Theorem 2.3.

Assume, by way of contradiction, that there are distinct $e_1, \ldots, e_n$ such that

$$COL(e_1) = \cdots = COL(e_n)$$

and

$$\sum_{i=1}^{n} a_i e_i = 0.$$

Let $\alpha_1, \ldots, \alpha_n$ be such that $e_i \in Z_{\alpha_i}$. Since all of the elements in any Z_α are colored differently, all of the α_i's are different. We will assume $\alpha_1 < \alpha_2 < \cdots < \alpha_n$. The other cases are similar. Note that

$$e_n = -(1/a_n) \sum_{i=1}^{n-1} a_i e_i \in CL(X)$$

and

$$e_1, \ldots, e_{n-1} \in Z_{\alpha_1} \cup \cdots \cup Z_{\alpha_{n-1}} \subseteq Y_{\alpha_{n-1}}.$$

Since $Y_{\alpha_{n-1}} = CL(X_{\alpha_{n-1}})$ and $e_1, \ldots, e_{n-1} \in Y_{\alpha_{n-1}}$, we have $e_n \in Y_{\alpha_{n-1}}$. Hence $e_n \notin Z_{\alpha_n}$. This is a contradiction. ∎

Note 2.6 For most linear equations, CH is not needed to get a counterexample.

3 $\neg$ CH $\Rightarrow$ TRUE

Theorem 3.1 *Assume CH is false. Let COL be an $\aleph_0$-coloring of* R. *There exist distinct e_1, e_2, e_3, e_4 such that*

$$COL(e_1) = COL(e_2) = COL(e_3) = COL(e_4),$$

$$e_1 + e_2 = e_3 + e_4.$$

Proof: By Fact 1.3 there is an injection of ω_2 into R. If $\alpha \in \omega_2$, then x_α is the real associated to it.

Let COL be an $\aleph_0$-coloring of R. We show that there exist distinct e_1, e_2, e_3, e_4 of the same color such that $e_1 + e_2 = e_3 + e_4$.

We define a map F from ω_2 to $\omega_1 \times \omega_1 \times \omega_1 \times \omega$.

1. Let $\beta \in \omega_2$.

2. Define a map from ω_1 to ω by

$$\alpha \mapsto COL(x_\alpha + x_\beta).$$

3. Let $\alpha_1, \alpha_2, \alpha_3 \in \omega_1$ be distinct elements of ω_1, and $i \in \omega$, such that $\alpha_1, \alpha_2, \alpha_3$ all map to i. Such $\alpha_1, \alpha_2, \alpha_3, i$ clearly exist since $\aleph_0 + \aleph_0 = \aleph_0 < \aleph_1$. (There are $\aleph_1$ many elements that map to the same element of ω, but we do not need that.)

4. Map β to $(\alpha_1, \alpha_2, \alpha_3, i)$.

Since F maps a set of cardinality $\aleph_2$ to a set of cardinality $\aleph_1$, there exists some element that is mapped to twice by F (actually there is an element that is mapped to $\aleph_2$ times, but we do not need this). Let $\alpha_1, \alpha_2, \alpha_3, \beta, \beta', i$ be such that $\beta \neq \beta'$ and

$$F(\beta) = F(\beta') = (\alpha_1, \alpha_2, \alpha_3, i).$$

Choose distinct $\alpha, \alpha' \in \{\alpha_1, \alpha_2, \alpha_3\}$ such that $x_\alpha - x_{\alpha'} \notin \{x_\beta - x_{\beta'}, x_{\beta'} - x_\beta\}$. We can do this because there are at least three possible values for $x_\alpha - x_{\alpha'}$.

Since $F(\beta) = (\alpha_1, \alpha_2, \alpha_3, i)$, we have

$$COL(x_\alpha + x_\beta) = COL(x_{\alpha'} + x_\beta) = i.$$

Since $F(\beta') = (\alpha_1, \alpha_2, \alpha_3, i)$, we have

$$COL(x_\alpha + x_{\beta'}) = COL(x_{\alpha'} + x_{\beta'}) = i.$$

Let

$$\begin{aligned} e_1 &= x_\alpha + x_\beta \\ e_2 &= x_{\alpha'} + x_{\beta'} \\ e_3 &= x_{\alpha'} + x_\beta \\ e_4 &= x_\alpha + x_{\beta'}. \end{aligned}$$

Then

$$COL(e_1) = COL(e_2) = COL(e_3) = COL(e_4)$$

and

$$e_1 + e_2 = e_3 + e_4.$$

Since $x_\alpha \neq x_{\alpha'}$ and $x_\beta \neq x_{\beta'}$, we have $\{e_1, e_2\} \cap \{e_3, e_4\} = \emptyset$.

Moreover, the equation $e_1 = e_2$ is equivalent to

$$x_\alpha - x_{\alpha'} = x_{\beta'} - x_\beta,$$

which is ruled out by our choice of α, α', and so $e_1 \neq e_2$.

Similarly, $e_3 \neq e_4$.

Thus e_1, e_2, e_3, e_4 are all distinct. ∎

Remark. All the results above hold practically verbatim with R replaced by R^k, for any fixed integer $k \geq 1$. In this more geometrical context, e_1, e_2, e_3, e_4 are vectors in k-dimensional Euclidean space, and the equation $e_1 + e_2 = e_3 + e_4$ says that e_1, e_2, e_3, e_4 are the vertices of a parallelogram (whose area may be zero).

4 More is Known!

To state the generalization of this theorem we need a definition.

Definition 4.1 An equation $E(e_1, \ldots, e_n)$ (e.g., $e_1 + e_2 = e_3 + e_4$) is *regular* if the following holds: *for all colorings $COL : \mathsf{R} \to \mathsf{N}$ there exists $\vec{e} = (e_1, \ldots, e_n)$ such that*

$$COL(e_1) = \cdots = COL(e_n),$$

$$E(e_1, \ldots, e_n),$$

and $e_1, \ldots, e_n$ are all distinct.

If we combine Theorems 2.3 and 3.1 we obtain the following.

Theorem 4.2 *$e_1 + e_2 = e_3 + e_4$ is regular iff $2^{\aleph_0} > \aleph_1$.*

Jacob Fox (2) has generalized this to prove the following.

Theorem 4.3 *Let $s \in \mathsf{N}$. The equation*

$$e_1 + se_2 = e_3 + \cdots + e_{s+3} \tag{1}$$

is regular iff $2^{\aleph_0} > \aleph_s$.

Fox's result also holds in higher dimensional Euclidean space, where it relates to the vertices of $(s+1)$-dimensional parallelepipeds. Subtracting $(s+1)e_2$ from both sides of (1) and rearranging, we get

$$e_1 - e_2 = (e_3 - e_2) + \cdots + (e_{s+3} - e_2),$$

which says that e_1 and e_2 are opposite corners of some $(s+1)$-dimensional parallelepiped P where $e_3, \ldots, e_{s+3}$ are the corners of P adjacent to e_2. Of course, there are other vertices of P besides these, and Fox's proof actually shows that if $2^{\aleph_0} > \aleph_s$ then *all* the 2^{s+1} vertices of some such P must have the same color.

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References

[1] R. O. Davies. Partioning the plane into denumerably many sets without repeated differences. *Proceedings of the Cambridge Philosophical Society*, 72:179–183, 1972.

[2] J. Fox. An infinite color analogue of Rado's theorem. `www.princeton.edu/~jacobfox/~publications.html`.

[3] P. Komjáth. Partitions of vector spaces. *Periodica Mathematica Hungarica*, 28:187–193, 1994.