The Solution to a Problem in a Romanian Math Problem Book

By Bill Gasarch

The following problem is in a Romanian book of math problems. It was told to me by Ioana Bercea who is a Romanian. The answers were in the book but in Romanian so I was forced to solve it myself.

Def 0.1 Let $Q(n)$ be the statement $(\exists x_1, \ldots, x_n \in \mathsf{N})[\sum_{i=1}^{n} \frac{1}{x_i^2} = 1]$.

Problem: Prove that for all $n \geq 6$ $Q(n)$ is true.

Plan: We will prove (not in this order) (I) $Q(6), Q(7), Q(8)$ and (II) $(\forall n)[Q(n) \implies Q(n+3)]$.

Def 0.2 Let $P(n, s, t_e, t_o)$ be YES if there exists a multiset $\{x_1, \ldots, x_n\}$ such that

- $\sum_{i=1}^{n} \frac{1}{x_i^2} = 1$.
- $\{x_1, \ldots, x_n\} = A_1 \cup \cdots \cup A_s \cup L_e \cup L_o$ where all of these multisets are disjoint, each A_i has four of the same even number in them, L_e contains t_e even numbers, L_o contains t_o odd numbers.

Note that we have $P(1, 0, 0, 1)$ via one 1.

Lemma 0.3

1. *If* $t_o \geq 1$ *then* $P(n, s, t_e, t_o) \implies P(n+3, s+1, t_e, t_o - 1)$.
2. *If* $t_e \geq 1$ *then* $P(n, s, t_e, t_o) \implies P(n+3, s+1, t_e, t_o)$.
3. *If* $s \geq 1$ *then* $P(n, s, t_e, t_o) \implies P(n+3, s, t_e + 3, t_o)$.
4. $(\forall n \geq 1)[Q(n) \implies Q(n+3)]$ *(This follows from the first three.)*

Proof: 1, (2,3): Replace an $x \in L_o$ ($x \in L_e$, $x \in A_1 \cup \cdots \cup A_s$) with $\{2x, 2x, 2x, 2x\}$. ∎

By Lemma 0.3 and $P(1,0,0,1)$ we get $P(4,1,0,0)$ and then $P(7,1,3,0)$. Hence we get $Q(7)$. We do this explicitly.

$P(1,0,0,1)$ via one 1.

$P(4,1,0,0)$ via four 2's

$P(7,1,3,0)$ via three 2's and four 4's

Lemma 0.4

1. *If* $t_e \geq 1$ *then* $P(n,s,t_e,t_o) \implies P(n+8,s+2,t_e,t_o)$.
2. *If* $t_o \geq 1$ *then* $P(n,s,t_e,t_o) \implies P(n+8,s,t_e,t_o+8)$.
3. *If* $s \geq 1$ *then* $P(n,s,t_e,t_o) \implies P(n+8,s+1,t_e+4,t_o)$.

Proof: 1, (2,3): Replace an $x \in L_e$ ($x \in L_o$, $x \in A_1 \cup \cdots \cup A_s$) with $\{3x,3x,3x,3x,3x,3x,3x,3x,3x\}$. ∎

By Lemma 0.4.3 and $P(4,1,0,0)$ we obtain $P(12,2,4,0)$. Then use Lemma 0.4.1 to obtain $P(20,4,4,0)$. We do this explicitly.

$P(4,1,0,0)$ via four 2's.

$P(12,2,4,0)$ via three 2's and nine 6's.

$P(20,4,4,0)$ via three 2's and eight 6's and nine 18's.

Lemma 0.5 *If* $s \geq 1$ *then* $P(n,s,t_e,t_o) \implies P(n-3,s-1,t_e',t_o')$ *where exactly one of* t_e'*,* t_o' *is one more than it was and the other stays the same.*

Proof: Replace $A_1 = \{x,x,x,x\}$ with $\{\frac{x}{2}\}$. (Recall that the A_i's have all even elements.) If $\frac{x}{2}$ is even then t_e increases by one. If $\frac{x}{2}$ is odd then t_o increases by one. ∎

Apply Lemma 0.5 four times to $P(20,4,4,0)$ to obtain $P(8,0,4,0)$, so we have $Q(8)$. We do this explicitly.

$P(20,4,4,0)$ via three 2's and eight 6's and nine 18's.

$P(17,3,4,0)$ via three 2's and eight 6's and one 9 and five 18's

$P(14,2,4,0)$ via three 2's and eight 6's and two 9's and one 18.

$P(11,1,4,0)$ via three 2's and one 3 and five 6's and two 9's and one 18.

$P(8,0,4,0)$ via three 2's and two 3's and one 6 and two 9's and one 18.

Apply Lemma 0.5 twice to $P(12,2,4,0)$ to obtain $P(6,0,4,0)$ so we have $Q(6)$.

$P(12,2,4,0)$ via three 2's and nine 6's.

$P(9,1,4,0)$ via three 2's and one 3 and five 6's.

$P(6,0,4,0)$ via three 2's and two 3 and one 6.

We have $Q(6), Q(7), Q(8)$ and $(\forall n \geq 1)[Q(n) \implies Q(n+3)]$. Hence we have $(\forall n \geq 6)[Q(n)]$.

Some notes.

1. The solution in the back of the book just gave the numbers to prove $Q(6), Q(7), Q(8)$ and proved $Q(n) \implies Q(n+3)$. There numbers were

   - $Q(6)$: three 2's two 3's and one 6. Same as mine.
   - $Q(7)$: three 2's and four 4's. Same as mine.
   - $Q(8)$: three 2's, two 3's, one 7, one 14, one 21. Different from mine.

   They do not say how they got it.

2. $Q(5)$ is false by a case by case analysis: You must use AT LEAST three 2's since if you used two 2's and three 3's then you get $2 \times \frac{1}{4} + 3 \times \frac{1}{9} < 1$. Hence we need (a,b) such that $\frac{1}{a^2} + \frac{1}{b^2} = 1 - \frac{3}{4} = \frac{1}{4}$. We leave it to the reader to show this cannot be done.

3. Note that the theorem with 6 is optimal.

We can prove a more general theorem but without stating the starting point.

Def 0.6 Let $k \in \mathsf{N}$. Let $Q_k(n)$ be the statement $(\exists x_1, \ldots, x_n \in \mathsf{N})[\sum_{i=1}^{n} \frac{1}{x_i^k} = 1]$.

Theorem 0.7 *For all k there exists n_o such that for all $n \geq n_o$ $Q_k(n)$ is true.*

Proof: Note that $Q_k(1)$ is true as $1 = \frac{1}{1^k}$.

Let $i \in \mathsf{N}$. Clearly $Q_k(n) \implies Q(n + i^k - 1)$: replace $\frac{1}{a_n^k}$ with i^k copies of $\frac{1}{(ia_n)^k}$. Hence for all $x_2, \ldots, x_m$ (any m), if

$$n = 1 + (2^k - 1)x_2 + (3^k - 1)x_3 + \cdots + (m^k - 1)x_k$$

then we have $Q(n)$. It is well known that if $a_1, a_2, \ldots, a_k$ are rel prime then almost all natural numbers can be written as a linear combination of them with positive coefficients. Hence we need to show that some subset of $\{2^k - 1, 3^k - 1, \ldots\}$ is rel prime. Let $d = GCD(2^k - 1, 3^k - 1)$. If $d = 1$ then you are done. If $d \geq 2$ then $GCD(2^k - 1, 3^k - 1, d^k - 1) = 1$ and we are done.

Alternative: $GCD(2^k - 1, 2^{2^k - 1} - 1) = 1$. ∎

Open Questions

1. Obtain upper and lower bounds on n_o as a function of k from the last theorem.

2. How hard is the following problem: Given (k, n) determine if 1 can be written a the sum of n inverses-kth-powers. If yes then produce a way to do this. (Greedy does not work— it fails for $k = 2$, $n = 8$.)

3. How hard is the following problem: Given (k, n) determine how many ways 1 can be written as the sum of n inverses-kth-powers.