A Complete High School Proof of Schur's Theorem on Making Change of n cents

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1 Introduction

Let $a_1, a_2, \ldots, a_L$ be coin denominations. Assume you have an unlimited number of each coin. How many ways can you make n cents with these coins? Schur's theorem gives the answer asymptotically and also yields the coefficient of the dominant term. We state it:

Theorem 1.1 *Let $a_1 < \cdots < a_L \in \mathsf{N}$ be relatively prime. Think of them as coin denominations. As n goes to infinity the number of ways to make change of n cents is $\frac{n^{L-1}}{(L-1)!a_1a_2\cdots a_L} + O(n^{L-2})$.*

I volunteered to present a proof of Schur's theorem to high school students taking pre-calculus. I then realized that the proof of Schur's theorem I knew, from Wilf's wonderful book on generating functions [1], uses generating function, roots of unity, partial fractions, and the Taylor series for $\frac{1}{(1-x)^L}$.

The material on generating functions and roots of unity is such that I could prove and use just what was needed. Partial fractions could certainly be taught to high school students; however, I wanted to prove they worked, not show them a cook book way to decompose. I then realized that

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I had never seen a proof that they worked; however, I doubt this would be hard to obtain. Getting a Taylor series without calculus for $\frac{1}{(1-x)^L}$ seemed like an interesting challenge.

I succeeded in my goals. We present a complete high school proof of Schur's theorem, essentially Wilf's proof, with the following features:

1. We prove the partial fractions decomposition that we need. The result is standard; however, our proof is short and makes our treatment self contained. We doubt our proof is new though we have not been able to find a reference.

2. We obtain the Taylor series of $\frac{1}{(1-x)^L}$ without calculus. We doubt our proof is new though we have not been able to find a reference.

3. We obtain that for all $1 \le r \le M-1$ where $M = LCM(a_1, \ldots, a_L)$ there is a polynomial h_r of degree $L-1$ such that if $n \equiv r \pmod{M}$ then $h_r(n)$ is the number of ways to make change of n. We doubt the result is new though we have not been able to find a reference.

2 Induction Proof of Partial Fractions Decomposition

Lemma 2.1

1. *For all $n \in \mathsf{N}$, for all $c, d \in \mathsf{C}$, $c \neq d$, there exists $A, A_1, \ldots, A_n$ such that*

$$\frac{1}{(1-cx)(1-dx)^n} = \frac{A}{1-cx} + \sum_{k=1}^{n} \frac{A_k}{(1-dx)^k}.$$

2. *For all $n_1, \ldots, n_L \in \mathsf{N}$, for all $c_1, \ldots, c_L \in \mathsf{C}$, distinct complex numbers, there exists $A_{i,j}$ such that*

$$\prod_{i=1}^{L} \frac{1}{(1-c_i x)^{n_i}} = \sum_{i=1}^{L} \sum_{j=1}^{n_i} \frac{A_{i,j}}{(1-c_i x)^j}.$$

Proof:

1) We prove this by induction on n.

Base Case: $n = 1$**.** We need to solve for A, A_1 in this equation:

$\frac{1}{1-cx}\frac{1}{1-dx} = \frac{A}{1-cx} + \frac{A_1}{1-dx}$.

$1 = A(1 - dx) + A_1(1 - cx)$, so $1 = (A + A_1) - (dA + cA_1)x$. Hence

$A + A_1 = 1$ and $dA + cA_1 = 0$. These can be easily solved to yield $A = \frac{c}{c-d}$ and $A_1 = \frac{-d}{c-d}$. Note that we are using $c \neq d$.

Induction Hypothesis (IH): We assume the lemma is true for $n - 1$.

By the IH there exists $A', A_2, \ldots, A_n$ (we purposely make these off by one so that later they will be what we want) such that

$$\frac{1}{(1-cx)(1-dx)^{n-1}} = \frac{A'}{1-cx} + \sum_{k=1}^{n-1} \frac{A_{k+1}}{(1-dx)^k}.$$

Hence

$$\begin{aligned}
\tfrac{1}{(1-cx)(1-dx)^n} &= \tfrac{1}{1-dx}\left[\tfrac{A'}{1-cx} + \textstyle\sum_{k=1}^{n-1} \tfrac{A_{k+1}}{(1-dx)^k}\right] = \tfrac{A'}{(1-cx)(1-dx)} + \textstyle\sum_{k=2}^{n} \tfrac{A_k}{(1-dx)^k} \\
&= \tfrac{A}{1-cx} + \tfrac{A_1}{1-dx} + \textstyle\sum_{k=2}^{n} \tfrac{A_k}{(1-dx)^k} \quad \text{(by the } n = 1 \text{ case)} \\
&= \tfrac{A}{1-cx} + \textstyle\sum_{k=1}^{n} \tfrac{A_k}{(1-dx)^j}
\end{aligned}$$

2) We prove this by induction on $\sum_{i=1}^{L} n_i$.

Base Case: $\sum_{i=1}^{L} n_i = 1$. This only happens when $L = 1$ and $n_1 = 1$ which is trivial.

Induction Hypothesis (IH): Assume the lemma is true for all $(n_1', \ldots, n_L')$ with $\sum_{i=1}^{L} n_i' < \sum_{i=1}^{L} n_i$.

Clearly

$$\prod_{i=1}^{L} \frac{1}{(1-c_i x)^{n_i}} = \frac{1}{1-c_1 x}\left[\frac{1}{(1-c_1 x)^{n_1-1}} \prod_{i=2}^{L} \frac{1}{(1-c_i)^{n_i}}\right]$$

We rewrite what is in the square brackets as

$$\frac{1}{(1-c_1x)^{n_1-1}}\frac{1}{(1-c_2)^{n_2}}\frac{1}{(1-c_3)^{n_3}}\cdots\frac{1}{(1-c_L)^{n_L}}$$

This is in the exact form of the lemma we are proving though note that the sum of the exponents is $(\sum_{i=1}^{L} n_i) - 1 < \sum_{i=1}^{L} n_i$. Hence by the IH there exists $A_{1,j+1}$ (we purposely make these off by one so that later they will be what we want) and $A'_{i,j}$ such that the expression in square brackets is the following (we seperate out the first term for notationaly convinence).

$$\sum_{j=1}^{n_1-1}\frac{A_{1,j+1}}{(1-c_1x)^j}+\sum_{i=2}^{L}\sum_{j=1}^{n_i}\frac{A'_{i,j}}{(1-c_ix)^j}$$

Hence our original product, $\prod_{i=1}^{L}\frac{1}{(1-c_ix)^{n_i}}$, is

$$\frac{1}{1-c_1x}\left[\sum_{j=1}^{n_1-1}\frac{A_{1,j+1}}{(1-c_1x)^j}+\sum_{i=2}^{L}\sum_{j=1}^{n_i}\frac{A'_{i,j}}{(1-c_ix)^j}\right]=\left[\sum_{j=1}^{n_1-1}\frac{A_{1,j+1}}{(1-c_1x)^{j+1}}+\sum_{i=2}^{L}\sum_{j=1}^{n_i}\frac{A'_{i,j}}{(1-c_ix)^j}\right]$$

We can re-index the first summation to get:

$$=\sum_{j=2}^{n_1}\frac{A_{1,j}}{(1-c_1x)^j}+\sum_{i=2}^{L}\sum_{j=1}^{n_i}\frac{A'_{i,j}}{(1-c_1x)(1-c_ix)^j}$$

By Part 1 there exists constants $A''_{i,j}$ and $A_{i,j,k}$ such that

$$\sum_{i=2}^{L}\sum_{j=1}^{n_i}\frac{A'_{i,j}}{(1-c_1x)(1-c_ix)^j}=\sum_{i=2}^{L}\sum_{j=1}^{n_i}\frac{A''_{i,j}}{1-c_1x}+\sum_{i=2}^{L}\sum_{j=1}^{n_i}\sum_{k=1}^{j}\frac{A_{i,j,k}}{(1-c_ix)^k}$$

Let $\sum_{i=2}^{L}\sum_{j=1}^{n_1} A''_{i,j} = A_{1,1}$. Then

$$\sum_{i=2}^{L}\sum_{j=1}^{n_1}\frac{A''_{i,j}}{1-c_1x}=\frac{A_{1,1}}{1-c_1x}.$$

For $2 \leq i \leq L$ and $1 \leq j \leq n_i$ let $A_{i,j} = \sum_{k=1}^{n_i} \sum_{j=k}^{n_i} A_{i,j,k}$. Then

$$\sum_{i=2}^{L} \sum_{j=1}^{n_i} \sum_{k=1}^{j} \frac{A_{i,j,k}}{(1-c_i x)^k} = \sum_{i=2}^{L} \sum_{k=1}^{n_i} \sum_{j=k}^{n_i} \frac{A_{i,j,k}}{(1-c_i x)^k} = \sum_{i=2}^{L} \sum_{k=1}^{n_i} \frac{A_{i,j}}{(1-c_i x)^k}$$

Hence our original product, $\prod_{i=1}^{L} \frac{1}{(1-c_i x)^{n_i}}$, is

$$\sum_{j=2}^{n_1} \frac{A_{1,j}}{(1-c_1 x)^j} + \frac{A_{1,1}}{1-c_1 x} + \sum_{i=2}^{L} \sum_{k=1}^{n_i} \frac{A_{i,j}}{(1-c_i x)^k} = \sum_{i=1}^{L} \sum_{j=1}^{n_i} \frac{A_{i,j}}{(1-c_i x)^j}.$$

∎

The usual theorem about partial fraction decomposition that is used in calculus starts with a polynomial over the reals and factors it into linear and quadratic polynomials over the reals. This version can easily be derived from Lemma 2.1

3 Non Calculus Proof of the Taylor Series for $\frac{1}{(1-x)^n}$

We obtain the Taylor expansion for $\frac{1}{(1-x)^L}$ via combinatorics, not calculus.

Def 3.1

1. If $n \in \mathsf{N}$ then $[n]$ is the set $\{1, \ldots, n\}$.
2. An *L-set of X* is a subset of X of size L.

Lemma 3.2 *For all n, L, $\sum_{i=0}^{n} \binom{L-1+i}{L-1} = \binom{L+n}{L}$.*

Proof: The term L-set will mean L-set of $\{1, \ldots, L+n\}$ throughout.

We solve the following problem two ways: How many L-sets are there? Clearly the answer is $\binom{L+n}{L}$.

Another way to solve this problem is to partition the L-sets based on the set's largest element. The largest element in any L-set is of the form $L+i$ where $0 \leq i \leq n$. The number of L-sets with largest element $L+i$ is the number of $(L-1)$-sets of $\{1,\ldots,L-1+i\}$, namely $\binom{L-1+i}{L-1}$. Hence the number of L-sets is $\sum_{i=0}^{n}\binom{L-1+i}{L-1}$. This yields our result. ∎

Lemma 3.3 *For all L, $\frac{1}{(1-x)^L}=\sum_{n=0}^{\infty}\binom{L-1+n}{L-1}x^n$.*

Proof: We prove this by induction on L.

Base Case: $L=1$**.** This is the well known geometric series $\frac{1}{1-x}=\sum_{i=0}^{\infty}x^i$.

Induction Hypothesis (IH): Assume the lemma is true for $L-1$:

$$\frac{1}{(1-x)^{L-1}}=\sum_{i=0}^{\infty}\binom{L-2+i}{L-2}x^i$$

From the IH we obtain:

$$\frac{1}{(1-x)^L}=\frac{1}{(1-x)^{L-1}}\frac{1}{1-x}=\left(\sum_{i=0}^{\infty}\binom{L-2+i}{L-2}x^i\right)\left(\sum_{j=0}^{\infty}x^j\right)$$

$$=\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\binom{L-2+i}{L-2}x^{i+j}=\sum_{n=0}^{\infty}\left(\sum_{i=0}^{n}\binom{L-2+i}{L-2}\right)x^n=\sum_{n=0}^{\infty}\binom{L-1+n}{L-1}x^n$$

This last equality used Lemma 3.2 ∎

4 Lemma about Roots of Unity

Lemma 4.1 *If $0\le x\le y\le 2\pi$, $\cos(x)=\cos(y)$, and $\sin(x)=\sin(y)$ then $x=y$.*

Proof: Since $\cos(x)=\cos(y)$ either $x=y$ or $x+y=2\pi$. Since $\sin(x)=\sin(y)$ either $x=y$ or $x+y\in\{\pi,3\pi\}$. Since $2\pi\notin\{\pi,3\pi\}$, $x=y$. ∎

Lemma 4.2 *Let $a_1 < \cdots < a_L \in \mathsf{N}$ be relatively prime. Let $g(x) = (x^{a_1} - 1) \cdots (x^{a_L} - 1)$. When $g(x)$ is factored completely into linear terms the factor $(x - 1)$ occurs L times and all of the other linear factors occur $\leq L - 1$ times.*

Proof: Clearly $x - 1$ occurs in all L of the polynomials $(x^{a_i} - 1)$ and hence occurs L times. Each polynomial $(x^{a_i} - 1)$ has distinct roots, so if another linear term occurs L times it has to occur as a factor in each $(x^{a_i} - 1)$.

Assume that there exists $\omega \neq 1$ such that $(x - \omega)$ divides each $(x^{a_i} - 1)$. We will show that $a_1, \ldots, a_L$ have a nontrivial common factor and hence are not relatively prime. For all $1 \leq i \leq L$ let ω_i be the primitive ith root of unity. For all i, since $x - \omega$ divides $x^{a_1} - 1$, ω is an a_ith root of unity. In particular there exists $1 \leq A \leq a_1 - 1$ such that $\omega_1^A = \omega$. Since $A \leq a_1 - 1$, a_1 does not divide A. Hence there is some prime power p^c that divides a_1 but does not divide A.

Let $2 \leq i \leq L$. We show that p divides a_i. Since ω is an a_ith root of unity there exists $1 \leq B \leq a_i - 1$ such that $\omega_1^A = \omega = \omega_i^B$. Hence

$$\begin{aligned} \omega_1^A &= \omega_i^B \\ \cos\tfrac{2\pi A}{a_1} + \sqrt{-1}\sin\tfrac{2\pi A}{a_1} &= \cos\tfrac{2\pi B}{a_i} + \sqrt{-1}\sin\tfrac{2\pi B}{a_i} \end{aligned}$$

Hence

$$\begin{aligned} \cos\tfrac{2\pi A}{a_1} &= \cos\tfrac{2\pi B}{a_i} \\ \sin\tfrac{2\pi A}{a_1} &= \sin\tfrac{2\pi B}{a_i} \end{aligned}$$

By Lemma 4.1 $A/a_1 = B/a_i$. Therefore $Aa_i = Ba_1$, hence a_1 must divide Aa_i. Since p^c divides a_1 but not A, p must divide a_i. ∎

5 Schur's Theorem

Theorem 5.1 *Let $a_1 < \cdots < a_L \in \mathsf{N}$ be relatively prime. Think of them as coin denominations.*

1. *Let $M = LCM(a_1, \ldots, a_L)$. Let $0 \leq r \leq M - 1$. There is a polynomial h_r of degree $L - 1$*

such that if $n \equiv r \pmod{M}$ then $h_r(n)$ is the number of ways to make change of n cents.

2. *For all $0 \leq r \leq M-1$ the coefficient of of n^{L-1} in h_r is $\frac{1}{(L-1)!a_1a_2\cdots a_L}$. Note that the coefficient does not depend on r.*

3. *(This follows from Parts 1 and 2.) The number of ways to make change of n cents is*

$$\frac{n^{L-1}}{(L-1)!a_1a_2\cdots a_L} + O(n^{L-2}).$$

Proof:

We get a formula for the number of ways to make change of n cents and then prove Parts 1 and 2. Part 3 follows from Parts 1 and 2.

The number of ways to make change of n cents is the coefficient of x^n in

$$\begin{aligned} f(x) &= (1 + x^{a_1} + x^{2a_1} + \cdots)(1 + x^{a_2} + x^{2a_2} + \cdots)\cdots(1 + x^{a_L} + x^{2a_L} + \cdots) \\ &= \tfrac{1}{(1-x^{a_1})(1-x^{a_2})\cdots(1-x^{a_L})} \end{aligned}$$

For all $1 \leq i \leq L$, $1 \leq j \leq a_i - 1$, let $\alpha_{i,j}$ be the jth a_ith roots of unity (we think of 1 as being the 0th root of unity). Formally $\alpha_{i,j} = \cos\frac{2\pi j}{a_i} + \sqrt{-1}\sin\frac{2\pi j}{a_i}$. Let $n_{i,j}$ be the number of times the factor $(1 - \alpha_{i,j}x)$ appears in $(1 - x^{a_1})(1 - x^{a_2})\cdots(1 - x^{a_L})$. By Lemma 4.2 $n_{i,j} \leq L - 1$. We rewrite $f(x)$ and use Lemma 2.1 and 3.3 to obtain

$$f(x) = \frac{1}{(1-x)^L\prod_{i=1}^{L}\prod_{j=1}^{a_i-1}(1-\alpha_{i,j}x)^{n_{i,j}}} = \sum_{i=1}^{L}\frac{A_i}{(1-x)^i} + \sum_{i=1}^{L}\sum_{j=1}^{a_i-1}\sum_{k=1}^{n_{i,j}}\frac{A_{i,j,k}}{(1-\alpha_{i,j}x)^k}.$$

$$= \sum_{i=1}^{L}\sum_{n=0}^{\infty}A_i\binom{i-1+n}{i-1}x^n + \sum_{i=1}^{L}\sum_{j=1}^{a_i-1}\sum_{k=1}^{n_{i,j}}\sum_{n=0}^{\infty}A_{i,j,k}\binom{k-1+n}{k-1}\alpha_{i,j}^n x^n.$$

$$= \sum_{n=0}^{\infty} \left(\sum_{i=1}^{L} A_i \binom{i-1+n}{i-1} + \sum_{i=1}^{L} \sum_{j=1}^{a_i-1} \sum_{k=1}^{n_{i,j}} A_{i,j,k} \binom{k-1+n}{k-1} \alpha_{i,j}^n \right) x^n.$$

1) Since $\alpha_{i,j}$ is an ith root of unity, $\alpha_{i,j}^n = \alpha_{i,j}^{n \bmod M}$. Hence if $n \equiv r \pmod{M}$ then the coefficient of x^n in $f(x)$, which is the answer we seek, is

$$h_r(n) = \sum_{i=1}^{L} A_i \binom{i-1+n}{i-1} + \sum_{i=1}^{L} \sum_{j=1}^{a_i-1} \sum_{k=1}^{n_{i,j}} A_{i,j,k} \binom{k-1+n}{k-1} \alpha_{i,j}^r$$

Clearly this is a polynomial in n of degree $L-1$.

2) We need to find A_L.

$$\frac{1}{(1-x^{a_1})(1-x^{a_2})\cdots(1-x^{a_L})} = \sum_{i=1}^{L} \frac{A_i}{(1-x)^i} + \sum_{i=1}^{L} \sum_{j=1}^{a_i-1} \sum_{k=1}^{n_{i,j}} \frac{A_{i,j,k}}{(1-\alpha_{i,j}x)^k}.$$

Multiply both sides by $(1-x)^L$

$$\frac{(1-x)^L}{(1-x^{a_1})(1-x^{a_2})\cdots(1-x^{a_L})} = A_L + \sum_{i=1}^{L-1} A_i(1-x)^{L-i} + \sum_{i=1}^{L} \sum_{j=1}^{a_i-1} \sum_{k=1}^{n_{i,j}} \frac{A_{i,j,k}(1-x^L)}{(1-\alpha_{i,j}x)^k}.$$

The left hand side can be rewritten as

$$\frac{1}{(1+x+x^2+\cdots+x^{a_1-1})(1+x+x^2+\cdots+x^{a_2-1})\cdots(1+x+x^2+\cdots+x^{a_L-1})}$$

As x approaches 1 (from the left) the LHS approaches $\frac{1}{a_1 a_2 \cdots a_L}$ and the RHS approaches A_L. Hence $A_L = \frac{1}{a_1 a_2 \cdots a_L}$. Therefore

$$h_r(n) = \frac{(n+1)(n+2)\cdots(n+L-1)}{(L-1)!a_1a_2\cdots a_L} + \sum_{i=1}^{L-1} A_i \binom{i-1+n}{i-1} + \sum_{i=1}^{L}\sum_{j=1}^{a_i-1}\sum_{k=1}^{n_{i,j}} A_{i,j,k}\binom{k-1+n}{k-1}\alpha_{i,j}^r$$

Since all $n_{i,j} \leq L-1$

$$h_r(n) = \frac{n^{L-1}}{(L-1)!a_1a_2\cdots a_L} + O(n^{L-2}).$$

∎

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References

[1] H. Wilf. *Generatingfunctionology*. Academic Press, Waltham, MA, 1994.