1 The Background and the Problem

I am sure everything in this writeup is known. When I posted the question on my blog someone gave the answer below, though without proof.

Consider the following language:
1. There are first order variables that range over elements of the domain.
2. There is a symbol + and it obeys the usual axioms- commutative and associative.
3. We have all of the logical symbols: ∧, ∨, ¬, ∀, ∃. The ∀ and ∃ range over the domain.

We can define terms, formulas, and statements.

Definition 1.1 A term is any expression of the form \( x_1 + \cdots + x_n \) where the \( x_i \) are variables.

Definition 1.2 A formula is defined as follows.
1. If \( t_1 \) and \( t_2 \) are terms then \( (t_1 = t_2) \) is a formula.
2. If \( f_1 \) and \( f_2 \) are formulas then \( (f_1 \lor f_2), (f_1 \land f_2) \) and \( \neg f_1 \) are formulas.
3. If \( f(x) \) is a formula with free variable \( x \) then \( (\exists x)[f(x)] \) and \( (\forall x)[f(x)] \) are formulas.

Definition 1.3 A sentence is a formula without free variables.

We give a statement in the second order language of + that is true in \((R, +)\) but false in \((Q, +)\).

We first give it in English.

There exists sets \( A, B \) such that both \((A, +)\) and \((B, +)\) are groups but \( A \cap B = \{0\} \).

We now give this as statement in second order +. We need some subformulas first.

1. \( NT(A) \) be the formula

\[
(\exists x)(\exists y)[x \neq y \land x \in A \land y \in A].
\]

This says that \( A \) has at least two distinct elements in it.

2. Let \( Z(x) \) be the formula

\[
(\forall y)[x + y = y].
\]

This says that \( x = 0 \). Note that \((\exists x)[Z(x)]\) is true in both \( R \) and \( Q \) and in both cases the \( x \) is 0.
3. Let $ZI(A, B)$ be the formula

$$(\forall x)((x \in A \land x \in B) \implies Z(x)).$$

This says that the only element in $A \cap B$ is 0.

4. Let $CL(A)$ be the formula

$$(\forall x)(\forall y)((x \in A \land y \in A) \implies x + y \in A].$$

This says that $A$ is closed under addition.

5. Let $INV(A)$ be the formula

$$(\forall x)(\exists y)(x \in A \implies Z(x + y])$$

This says that $A$ is closed under additive inverses.

6. Let $GR(A)$ be the formula

$$CL(A) \land INV(A) \land NT(A).$$

This says that $A$ is a group with at least two elements.

**Theorem 1.4** Let $\psi$ be the following sentence in the second order language of $+$. 

$$\psi = (\exists A)(\exists B)[GR(A) \land GR(B) \land ZI(A, B)].$$

Then 

1. $(\mathbb{R}, +) \models \phi,$

2. $(\mathbb{Q}, +) \models \neg \phi.$

**Proof:**

We first show that the statement is true in $\mathbb{R}$. 

Let

$$A = \{q\pi \mid q \in \mathbb{Q}\}.$$ 

$$B = \mathbb{Q}.$$ 

Clearly both $A$ and $B$ are groups with at least two elements in them. One can easily show that if $x \in A \cap B$ then $x = 0$ (else $\pi \in \mathbb{Q}$). 

We now show that the statement is false in $\mathbb{Q}$. Assume, by way of contradiction, that the statement is true in $\mathbb{Q}$. Since $A$ and $B$ must have at least two elements, they each must have at least one
nonzero element. Since $A$ and $B$ are closed under additive inverses they must each have a positive element.

Let \( \frac{p_1}{q_1} \in A \cap \mathbb{Q}^+ \) and \( \frac{p_2}{q_2} \in B \cap \mathbb{Q}^+ \). Since $A$ is closed under addition, for all \( n_1 \in \mathbb{N} \), \( \frac{n_1 p_1}{q_1} \in A \). Since $B$ is closed under addition, for all \( n_2 \in \mathbb{N} \), \( \frac{n_2 p_2}{q_2} \in B \). Let \( n_1 = q_1 p_2 \) and \( n_2 = q_2 p_1 \). This yields that \( p_1 p_2 \in A \) and \( p_1 p_2 \in B \). Hence there is a nonzero element in $A \cap B$. This is a contradiction. 

Note that we have a statement of the form

\[ (\exists A)(\exists B) \text{ first order stuff } \]

that is true of $\mathbb{R}$ but not of $\mathbb{Q}$. Is there a statement of the form \( (\exists A)[(\exists A) \text{ first order stuff }] \) that is true of $\mathbb{R}$ but not of $\mathbb{Q}$. YES- we can state that there exists two groups that overlap only at 0 with just one second order quantifier.

Intuitively $A$ will be the union of the two groups. We will have \( x, y \in A \) such that \( x + y \notin A \) and then use

\[ A_x = \{ c \in A : x + c \in A \} \]

and

\[ A_y = \{ c \in A : y + c \in A \} \]

as our two subgroups.

We leave it to the reader to work out the exact sentence of the form \( (\exists A)[ \text{ first order stuff } ] \) that suffices.

2 Acknowledgments

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