Second Order Statements True in $(R, +)$ but not $(Q, +)$

1 The Background and the Problem

I am sure everything in this writeup is known. When I posted the question on my blog someone gave the answer below, though without proof.

Consider the following language:

1. There are first order variables that range over elements of the domain.
2. There is a symbol $+$ and it obeys the usual axioms- communative and associative.
3. We have all of the logical symbols: $\wedge, \vee, \neg, \forall, \exists$. The $\forall$ and $\exists$ range over the domain.

We can define terms, formulas, and statements.

Definition 1.1 A *term* is any expression of the form $x_1 + \cdots + x_n$ where the x_i are variables.

Definition 1.2 A formula is defined as follows.

1. If t_1 and t_2 are terms then $(t_1 = t_2)$ is a formula.
2. If f_1 and f_2 are formulas then $(f_1 \vee f_2)$, $(f_1 \wedge f_2)$ and $\neg f_1$ are formulas.
3. If $f(x)$ is a formula with free variable x then $(\exists x)[f(x)]$ and $(\forall x)[f(x)]$ are formulas.

Definition 1.3 A *sentence* is a formula without free variables.

We give a statement in the second order language of $+$ that is true in $(\mathsf{R}, +)$ but false in $(\mathsf{Q}, +)$. We first give it in English.

There exists sets A, B such that both $(A, +)$ and $(B, +)$ are groups but $A \cap B = \{0\}$.
We now give this is as statement in second order $+$. We need some subformulas first.

1. $NT(A)$ be the formula
$$(\exists x)(\exists y)[x \neq y \wedge x \in A \wedge y \in A].$$
This says that A has at least two distinct elements in it.
2. Let $Z(x)$ be the formula
$$(\forall y)[x + y = y].$$
This says that $x = 0$. Note that $(\exists x)[Z(x)]$ is true in both R and Q and in both cases the x is 0.

3. Let $ZI(A,B)$ be the formula

$$(\forall x)[(x \in A \wedge x \in B) \implies Z(x)].$$

   This says that the only element in $A \cap B$ is 0.

4. Let $CL(A)$ be the formula

$$(\forall x)(\forall y)[(x \in A \wedge y \in A) \implies x + y \in A].$$

   This says that A is closed under addition.

5. Let $INV(A)$ be the formula

$$(\forall x)(\exists y)[x \in A \implies Z(x + y)]$$

   This says that A is closed under additive inverses.

6. Let $GR(A)$ be the formula

$$CL(A) \wedge INV(A) \wedge NT(A).$$

   This says that A is a group with at least two elements.

Theorem 1.4 *Let ψ be the following sentence in the second order language of $+$.*

$$\psi = (\exists A)(\exists B)[GR(A) \wedge GR(B) \wedge ZI(A,B)].$$

Then

1. $(\mathsf{R}, +) \models \phi$,
2. $(\mathsf{Q}, +) \models \neg\phi$.

Proof:
We first show that the statement is true in R.
Let

$$A = \{q\pi \mid q \in \mathsf{Q}\}.$$

$$B = \mathsf{Q}.$$

Clearly both A and B are groups with at least two elements in them. One can easily show that if $x \in A \cap B$ then $x = 0$ (else $\pi \in \mathsf{Q}$).
We now show that the statement is false in Q. Assume, by way of contradiction, that the statement is true in Q. Since A and B must have at least two elements, they each must have at least one

nonzero element. Since A and B are closed under additive inverses they must each have a positive element.

Let $\frac{p_1}{q_1} \in A \cap \mathsf{Q}^+$ and $\frac{p_2}{q_2} \in B \cap \mathsf{Q}^+$. Since A is closed under addition, for all $n_1 \in \mathsf{N}$, $\frac{n_1 p_1}{q_1} \in A$. Since B is closed under addition, for all $n_2 \in \mathsf{N}$, $\frac{n_2 p_2}{q_2} \in B$. Let $n_1 = q_1 p_2$ and $n_2 = q_2 p_1$. This yields that $p_1 p_2 \in A$ and $p_1 p_2 \in B$. Hence there is a nonzero element in $A \cap B$. This is a contradiction. ∎

Note that we have a statement of the form

$$(\exists A)(\exists B) \text{ first order stuff]}$$

that is true of R but not of Q. Is there a statement of the form $(\exists A)[(\exists A)$ first order stuff] that is true of R but not of Q. YES- we can state that there exists two groups that overlap only at 0 with just one second order quantifier.

Intuitively A will be the union of the two groups. We will have $x, y \in A$ such that $x + y \notin A$ and then use

$$A_x = \{c \in A : x + c \in A\}$$

and

$$A_y = \{c \in A : y + c \in A\}$$

as our two subgroups.

We leave it to the reader to work out the exact sentence of the form $(\exists A)$[first order stuff] that suffices.

2 Acknowledgments

I would like to thank Chris Lastowski and James Pinkerton for helpful discussion.