When are both $x^2 + 3y$ and $y^2 + 3x$ Squares

Exposition by William Gasarch

1 Introduction

I was the following problem in a book of math contest problems. The solution was not provided and I was unable to do it, so I posted asking for a solution. I got two. I present the problem and both solutions, and what I found out by trying to solve the problem.

For which $(x, y)$, positive naturals, is both $x^2 + 3y$ and $y^2 + 3x$, squares?

2 My Failed Attempt but a Question About it

If $x^2 + 3y$ is a square then $x^2 + 3y \geq (x + 1)^2 = x^2 + 2x + 1$. Hence $y \geq \frac{2x+1}{3}$.

By symmetry we also get $x \geq \frac{2y+1}{3}$ which we rewrite as $\frac{3x-1}{2} \geq y$.

So we have $\frac{2x+1}{3} \leq y \leq \frac{3x-1}{2}$.

I then had a grad student, Daniel Smolyak, write a program look at all $(x, y)$ with $1 \leq x, y \leq 1000$ and $\frac{2x+1}{3} \leq y \leq \frac{3x-1}{2}$.

trying to find a case where $x^2 + 3y$ and $y^2 + 3x$ are both squares. He found three: (1,1), (11,16), (16,11).

We suspected these were the only pairs.

3 Solution provided by Marzio De Biasi

Let $a, b \geq 1$ be such that

\[ x^2 + 3y = (x + a)^2 = x^2 + 2ax + a^2, \text{ so } y = \frac{2ax + a^2}{3}. \]

\[ y^2 + 3x = (y + b)^2 = y^2 + 2bx + b^2, \text{ so } x = \frac{2by + b^2}{3}. \]

Take the equation for $y$ and put the $x$ in:

Lets be careful:

\[ 2ax + a^2 = 2a\left(\frac{2by + b^2}{3}\right) + a^2 = \frac{4aby + 2ab^2 + 3a^2}{3}. \]
SO

\[ y = \frac{2ax + a^2 + 3a^2}{3} = \frac{4aby + 2ab^2 + 3a^2}{9} \]

\[ 9y = 4aby + 2ab^2 + 3a^2 \]

\[ (9 - 4ab)y = 2ab^2 + 3a^2 \]

\[ y = \frac{2ab^2 + 3a^2}{9 - 4ab} \]

Since \( a, b \geq 1 \), the numerator is positive. Since \( y \geq 1 \) then denominator has to be positive. The only pairs of positive naturals \((a, b)\) such that \(9 - 4ab \geq 1\) are \((a, b) = (1, 1), (1, 2), (2, 1)\). We leave the rest to you.

4 Solution by xxx

Since this solution is by xxx I assume its anonymous.

Assume \( y \leq x \). Then

\[ 3y < 4x + 4 \] (an overestimation but you will soon see why we make it)

Hence

\[ x^2 + 3y < x^2 + 4x + 4 = (x + 2)^2. \]

The only square that \( x^2 + 3y \) can be is \((x + 1)^2\). So

\[ x^2 + 3y = (x + 1)^2 = x^2 + 2x + 1 \]
\[3y = 2x + 1\]

So \(y\) is odd, let \(y = 2a + 1\). Then \(x = 3a + 1\).

Since \(y^2 + 3x\) is a square the following is a square:

\[(2a + 1)^2 + 3(3a + 1) = 4a^2 + 13a + 4.\]

Note that
\[(2a + 2)^2 = 4a^2 + 8a + 4\] is small.
\[(2a + 4)^2 = 4a^2 + 16a + 16\] is too big.

Hence

\[4a^2 + 13a + 4 = (2a + 3)^2 = 4a^2 + 12a + 9\]

\(a = 5\). We leave the rest to you.

5 **The Question this Raises**

Is the following solvable: given \(p, q \in \mathbb{Z}[x, y]\)

1. Determine if there is an infinite number of \((x, y)\) such that \(p(x, y)\) and \(q(x, y)\) are both squares.

2. If there are a finite number of values find them. (Or find how many there are, which is equivalent.)

One can replace *square* with the image of any \(r \in \mathbb{Z}[x]\) (or even \(r \in \mathbb{Z}[x_1, \ldots, x_n]\)).