When are both $x^2 + 3y$ and $y^2 + 3x$ Squares

Exposition by William Gasarch

1 Introduction

I was the following problem in a book of math contest problems. The solution was not provided and I was unable to do it, so I posted asking for a solution. I got two. I present the problem and both solutions, and what I found out by trying to solve the problem.

For which (x, y), positive naturals, is both $x^2 + 3y$ and $y^2 + 3x$, squares?

2 My Failed Attempt but a Question About it

If $x^2 + 3y$ is a square then $x^2 + 3y \geq (x + 1)^2 = x^2 + 2x + 1$. Hence $y \geq \frac{2x+1}{3}$.

By symmetry we also get $x \geq \frac{2y+1}{3}$ which we rewrite as $\frac{3x-1}{2} \geq y$.

So we have $\frac{2x+1}{3} \leq y \leq \frac{3x-1}{2}$.

I then had a grad student, Daniel Smolyak, write a program look at all (x, y) with $1 \leq x, y \leq 1000$ and $\frac{2x+1}{3} \leq y \leq \frac{3x-1}{2}$.

trying to find a case where $x^2 + 3y$ and $y^2 + 3x$ are both squares. He found three: (1,1), (11,16), (16,11).

We suspected these were the only pairs.

3 Solution provided by Marzio De Biasi

Let $a, b \geq 1$ be such that

$x^2 + 3y = (x + a)^2 = x^2 + 2ax + a^2$, so $y = \frac{2ax+a^2}{3}$.

$y^2 + 3x = (y + b)^2 = y^2 + 2bx + b^2$, so $x = \frac{2by+b^2}{3}$.

Take the equation for y and put the x in:

Lets be careful:

$2ax + a^2 = 2a(\frac{2by+b^2}{3}) + a^2 = \frac{4aby+2ab^2+3a^2}{3}$.

SO

$$y = \frac{2ax + a^2 + 3a^2}{3} = \frac{4aby + 2ab^2 + 3a^2}{9}$$

$$9y = 4aby + 2ab^2 + 3a^2$$

$$(9 - 4ab)y = 2ab^2 + 3a^2$$

$$y = \frac{2ab^2 + 3a^2}{9 - 4ab}$$

Since $a, b \geq 1$, the numerator is positive. Since $y \geq 1$ then denominator has to be positive. The only pairs of positive naturals (a, b) such that $9 - 4ab \geq 1$ are

$(a, b) = (1, 1), (1, 2), (2, 1)$. We leave the rest to you.

4 Solution by xxx

Since this solution is by xxx I assume its anonymous.

Assume $y \leq x$. Then

$3y < 4x + 4$ (an overestimation but you will soon see why we make it)

Hence

$$x^2 + 3y < x^2 + 4x + 4 = (x + 2)^2.$$

The only square that $x^2 + 3y$ can be is $(x + 1)^2$. So

$$x^2 + 3y = (x + 1)^2 = x^2 + 2x + 1$$

$$3y = 2x + 1$$

So y is odd, let $y = 2a + 1$. Then $x = 3a + 1$.

Since $y^2 + 3x$ is a square the following is a square:

$$(2a + 1)^2 + 3(3a + 1) = 4a^2 + 13a + 4.$$

Note that

$(2a + 2)^2 = 4a^2 + 8a + 4$ is to small.

$(2a + 4)^2 = 4a^2 + 16a + 16$ is too big.

Hence

$$4a^2 + 13a + 4 = (2a + 3)^2 = 4a^2 + 12a + 9$$

$a = 5$. We leave the rest to you.

5 The Question this Raises

Is the following solvable: given $p, q \in \mathsf{Z}[x, y]$

1. Determine if there is an infinite number of (x, y) such that $p(x, y)$ and $q(x, y)$ are both squares.
2. If there are a finite number of values find them. (Or find how many there are, which is equivalent.)

One can replace *square* with the image of any $r \in \mathsf{Z}[x]$ (or even $r \in \mathsf{Z}[x_1, \ldots, x_n]$).