Steinitz's Lemma

An Exposition by William Gasarch

1 Introduction

This exposition is based on Imre Barany's article [1].

Consider the following easy theorem:

Theorem 1.1 *If* $V = \{v_1, \ldots, v_n\} \subseteq \mathsf{R}^1$, $(\forall i)[|v_i| \le 1]$, *and* $\sum_{i=1}^{n} v_i = 0$ *then there exists a reordering* $w_1, \ldots, w_n$ *of* V *such that* $(\forall k)[|\sum_{i=1}^{k} w_i| \le 1]$.

We will prove this in Section 2. How does this theorem generalize to R^2? to R^d? The absolute value signs now become magnitudes of vectors. Hence one might make the following conjecture:

Conjecture 1.2 *If* $V = \{v_1, \ldots, v_n\} \in \mathsf{R}^2$, $(\forall i)[|v_i| \le 1]$, *and* $\sum_{i=1}^{n} v_i = 0$ *then there exists a reordering* $w_1, \ldots, w_n$ *of* V *such that* $(\forall k)[|\sum_{i=1}^{k} w_i| \le 1]$.

Alas this is not true. Take the 3 third roots of unity. The sum of any 2 has magnitude bigger than 1. Hence we can never get $|w_1 + w_2| \le 1$. We leave it as an exercise to find, for all n, a counterexample.

What if we don't insist on 1 as an upper bound? What if we want to go to higher dimensions? The following are true:

1. If $V = \{v_1, \ldots, v_n\} \in \mathsf{R}^2$, $(\forall i)[|v_i| \le 1]$, and $\sum_{i=1}^{n} v_i = 0$ then there exists a reordering $w_1, \ldots, w_n$ of V such that $(\forall k)[|\sum_{i=1}^{k} w_i| \le \sqrt{5} \sim 2.24]$.
2. If $V = \{v_1, \ldots, v_n\} \in \mathsf{R}^d$, $(\forall i)[|v_i| \le 1]$, and $\sum_{i=1}^{n} v_i = 0$ then there exists a reordering $w_1, \ldots, w_n$ of V such that $(\forall k)[|\sum_{i=1}^{k} w_i| \le \sqrt{(4^d - 1)/3}]$.
3. If $V = \{v_1, \ldots, v_n\} \in \mathsf{R}^2$, $(\forall i)[|v_i| \le 1]$, and $\sum_{i=1}^{n} v_i = 0$ then there exists a reordering $w_1, \ldots, w_n$ of V such that $(\forall k)[|\sum_{i=1}^{k} w_i| \le 2]$.

4. If $V = \{v_1, \ldots, v_n\} \in \mathsf{R}^2$, $(\forall i)[|v_i| \le 1]$, and $\sum_{i=1}^{n} v_i = 0$ then there exists a reordering $w_1, \ldots, w_n$ of V such that $(\forall k)[|\sum_{i=1}^{k} w_i| \le d]$.

We will prove items 1 and 3. Item 2 has a similar proof as item 1, and Item 4 has a similar proof as item 3.

2 The $d = 1$ Case

To prove the $d = 1$ case we first prove a lemma which will be helpful in the $d = 2$ case.

Def 2.1 Let $A = (a_1, \ldots, a_m)$ and $B = (b_1, \ldots, b_L)$ An *interleaving of A and B* is an ordering on the elements $\{a_1, \ldots, a_m, b_1, \ldots, b_L\}$ such that if $i < j$ then a_i will proceed a_j and b_i will proceed b_j. As an example, if $m = 3$ and $L = 4$ then $(a_1, b_1, b_2, b_3, a_2, a_3, b_4)$ is an interleaving of A and B.

Lemma 2.2 *If $A = (a_1, \ldots, a_m) \in [0,1]^m$, $B = (b_1, \ldots, b_L) \in [-1,0]^L$, and $\sum_{x \in A \cup B} x = 0$, then there exists an interleaving of A and B, $(w_1, \ldots, w_{m+L})$ such that $(\forall k \le m+L)[|\sum_{i=1}^{k} w_i| \le 1]$.*

Proof: We define the reorder inductively.

Let

$$w_1 = a_1.$$

Assume that $w_1, \ldots, w_p$ are defined, $p < m + L$, and $(\forall k \le p)[|\sum_{i=1}^{k} w_i| \le 1]$.

1. **Case 1:** $\sum_{i=1}^{p} w_i < 0$. There must exist an element of A to draw from since otherwise $\sum_{x \in A \cup B} x < 0$. Let w_{p+1} be the next element of A. Clearly $|\sum_{i=1}^{p+1} w_i| \le 1$.

2. **Case 2:** $\sum_{i=1}^{p} w_i > 0$. There must exist an element of B to draw from since otherwise $\sum_{x \in A \cup B} x > 0$. Let w_{p+1} be the next element of B. Clearly $|\sum_{i=1}^{p+1} w_i| \le 1$.

3. **Case 3:** $\sum_{i=1}^{p} w_i = 0$. If there is an element of A available to take let w_{p+1} be the next element of A. If not then if there is an element of B available. Clearly $\left|\sum_{i=1}^{p+1} w_i\right| \leq 1$.

∎

Theorem 2.3 *If* $V = \{v_1, \ldots, v_n\} \subseteq \mathsf{R}^1$, $(\forall i)[|v_i| \leq 1]$, *and* $\sum_{i=1}^{n} v_i = 0$ *then there exists a reordering* $w_1, \ldots, w_n$ *of* V *such that* $(\forall k)[\left|\sum_{i=1}^{k} w_i\right| \leq 1]$.

Proof: Let A be the nonnegative elements in V. Let B be the negative elements in V. Apply Lemma 2.2. ∎

3 The $d = 2$ Case, First Proof

Theorem 3.1 *If* $V = \{v_1, \ldots, v_n\} \subseteq \mathsf{R}^2$, $(\forall i)[|v_i| \leq 1]$, *and* $\left|\sum_{i=1}^{n} v_i = 0\right.$ *then there exists a reordering* $w_1, \ldots, w_n$ *of* V *such that* $(\forall k)[\left|\sum_{i=1}^{k} w_i\right| \leq \sqrt{5}]$.

Proof:

Let $U \subseteq V$ be such that $\left|\sum_{u \in U} U\right|$ is maximized. Let $u^* = \sum_{u \in U} u$. We can assume that u^* is of the form $(0, a)$, so it points straight up.

The following facts are easily verified:

1. Every vector in V that is above or on the x-axis is in V. (If not then add that vector to U to form a U' with $\left|\sum_{u \in U} u\right| < \left|\sum_{u \in U'} u\right|$.)

2. The sum of the x-coordinates of the vectors in U is zero (since $u^* = (0, a)$.

3. Every vector in V that is below the x-axis is in $\overline{U}$. (If not then remove that vector from U to form a U' with $\left|\sum_{u \in U} u\right| < \left|\sum_{u \in U'} u\right|$.)

4. The sum of the x-coordinates of the vectors in $\overline{U}$ is zero (Since $\sum_{u \in U} u + \sum_{u \in \overline{U}} u = 0$ and $\sum_{u \in \overline{U}} = 0$.

Let $U = \{(a_1, b_1), \ldots, (a_m, b_m)\}$ and $\overline{U} = \{(c_1, d_1), \ldots, (c_L, d_L)\}$. By items 2 and 4 we have $\sum_{i=1}^{m} a_i = 0$ and $\sum_{i=1}^{L} c_i = 0$. Apply Theorem 2.3 to both $\{a_1, \ldots, a_m\}$ and $\{c_1, \ldots, c_L\}$. By renumbering we can assume that

- $(\forall k \le m)[\big|\sum_{i=1}^{k} a_i\big| \le 1]$, and
- $(\forall k \le L)[\big|\sum_{i=1}^{k} c_i\big| \le 1]$

By items 1 and 3 $(\forall i \le m)[b_i \ge 0]$ and $(\forall i \le L)[d_i \le 0]$. Since $\sum_{i=1}^{n} v_i = 0$ we know that $\sum_{i=1}^{m} b_i + \sum_{i=1}^{L} d_i = 0$. By Lemma 2.2 there is an interlacing $(b_1, \ldots, b_m)$ and $(d_1, \ldots, d_L)$ so that any initial partial sum has absolute value ≤ 1.

We use this ordering. Let it be

$$(x_1, y_1), \ldots, (x_n, y_n).$$

$$\sum_{i=1}^{k} (x_i, y_i) = (\sum_{i=1}^{k} x_i, \sum_{i=1}^{k} y_i).$$

Since $\big|\sum_{i=1}^{k} x_i\big|$ is the sum of a partial initial sums of the a_i's and of the b_i's by the triangle inequality this quantity is ≤ 2.

Since we used Lemma 2.2 $\big|\sum_{i=1}^{k} y_i\big| \le 1$.

Note that in the above few lines we used $\big|\cdots\big|$ to mean absolute value. We now use it to mean distance in R^2.

$$\big|\sum_{i=1}^{k} (x_i, y_i)\big| = \big|(\sum_{i=1}^{k} x_i, \sum_{i=1}^{k} y_i)\big| = \sqrt{(\sum_{i=1}^{k} x_i)^2 + (\sum_{i=1}^{k} y_i)^2} \le \sqrt{2^2 + 1^2} = \sqrt{5}.$$

∎

Using the same technique one can show the following:

Theorem 3.2 *Let $d \geq 1$. If $V = \{v_1, \ldots, v_n\} \subseteq \mathsf{R}^d$, $(\forall i)[|v_i| \leq 1]$, and $\sum_{i=1}^n v_i = 0$ then there exists a reordering $w_1, \ldots, w_n$ of V such that $(\forall k)[|\sum_{i=1}^k w_i| \leq \sqrt{\frac{4^d-1}{3}}]$.*

4 The $d = 2$ Case, Second Proof

Theorem 4.1 *If $V = \{v_1, \ldots, v_n\} \subseteq \mathsf{R}^2$, $(\forall i)[|v_i| \leq 1]$, and $\sum_{i=1}^n v_i = 0$ then there exists a reordering $w_1, \ldots, w_n$ of V such that $(\forall k)[|\sum_{i=1}^k w_i| \leq 2]$.*

Proof: Rewrite $\sum_{i=1}^n v_i = 0$ as

$$\sum_{i=1}^{n} \left(\frac{n-2}{n}\right) v_i = 0$$

and note that

$$\sum_{i=1}^{n} \frac{n-2}{n} = n-2.$$

We restate this in a less informative way in order to generalize it:

$\exists \alpha_1, \ldots, \alpha_n$ such that

- $(\forall i)[0 \leq \alpha_i \leq 1]$.
- $\sum_{i=1}^n \alpha_i v_i = 0$
- $\sum_{i=1}^n \alpha_i = n-2$

We want to derive re-order the v_i's (but we still call them $v_1, \ldots, v_n$), remove v_n from the list, and have the following:

$\exists \beta_1, \ldots, \beta_n$) such that

- $(\forall i)[0 \leq \beta_i \leq 1]$.
- $\sum_{i=1}^{n-1} \beta_i v_i = 0$
- $\sum_{i=1}^{n-1} n\beta_i = n-3$

We first look at n-tuples of β_i's and then see if we can make one of them 0. Let

$$TEMP = \{x \in [0,1]^n : \sum_{i=1}^{n} \beta_i v_i = 0 \wedge \sum_{i=1}^{n} \beta_i = n-3].\}$$

Note that

- $(\frac{n-3}{n-2}\alpha_1, \ldots, \frac{n-3}{n-2}\alpha_n) \in TEMP$. In particular $TEMP \neq \emptyset$.
- $TEMP$ is a convex polytope that is a subset of $[0,1]^n$.
- There is an $n \times 3$ matrix A and a vector $b \in \mathsf{R}^3$ such that

$$TEMP = \{(\beta_1, \ldots, \beta_n) \in [0,1]^n : Ax \leq b].\}$$

We need a Lemma

Lemma 4.2 *Let A be an $n \times e$ matrix and $b \in \mathsf{R}^e$. Let*

$$B = \{x \in [0,1]^n : Ax = b\}.$$

If B is nonempty then there exists a point in B with $\geq n-e$ of the variables in $\{0,1\}$

Proof sketch: Take an extreme point of B. ▮

$TEMP$ satisfies the condition of B in Lemma 4.2 with $e = 3$. Hence there is a point in $TEMP$ with $\geq n-3$ coordinates in $\{0,1\}$. One of the equations is

$$\beta_1 + \cdots + \beta_n = n-3.$$

Hence there must be an i with $\beta_i = 0$. Reorder to make that β_n. We are done.

We can keep doing this to obtain, for all $1 \leq k \leq n$, there exists $\beta_1, \ldots, \beta_i$ with

- $\sum_{i=1}^{k} \beta_i v_i = 0$
- $\sum_{i=1}^{k} \beta_i = k - 2$

This give us a reordering of $v_1, \ldots, v_n$ (though we still call it $v_1, \ldots, v_n$). We are concerned with $\left| \sum_{i=1}^{k} v_i \right|$.

First note that

$$\sum_{i=1}^{k} v_i = \sum_{i=1}^{k} v_i - \sum_{i=1}^{k} \beta_i v_i = \sum_{i=1}^{k} (1 - \beta_i) v_i.$$

Hence

$$\left| \sum_{i=1}^{k} v_i \right| = \left| \sum_{i=1}^{k} (1-\beta_i) v_i \right| \leq \sum_{i=1}^{k} \left|1-\beta_i\right| \left|v_i\right| \leq \sum_{i=1}^{k} \left|1-\beta_i\right| = \sum_{i=1}^{k} 1-\beta_i = k - \sum_{i=1}^{k} \beta_i = k-(k-2)$$

$$= 2.$$

∎

Using the same technique one can show the following:

Theorem 4.3 *Let $d \geq 1$. If $V = \{v_1, \ldots, v_n\} \subseteq \mathsf{R}^d$, $(\forall i)[\left|v_i\right| \leq 1]$, and $\sum_{i=1}^{n} v_i = 0$ then there exists a reordering $w_1, \ldots, w_n$ of V such that $(\forall k)[\left| \sum_{i=1}^{k} w_i \right| \leq d$.*

References

[1] I. Barany. On the power of linear depenencies. In *Building bridges between math and computer science (Bolyai society mathematical studies number 19)*, pages 31–45. Springer, 2008. `www.renyi.hu/~barany/cikkek/steinitz.pdf`.