Steinitz’s Lemma
An Exposition by William Gasarch

1 Introduction

This exposition is based on Imre Barany’s article [1].

Consider the following easy theorem:

**Theorem 1.1** If \( V = \{v_1, \ldots, v_n\} \subseteq \mathbb{R}^1 \), \((\forall i)|v_i| \leq 1\), and \( \sum_{i=1}^{n} v_i = 0 \) then there exists a reordering \( w_1, \ldots, w_n \) of \( V \) such that \((\forall k)|\sum_{i=1}^{k} w_i| \leq 1\).

We will prove this in Section 2. How does this theorem generalize to \( \mathbb{R}^2 \)? to \( \mathbb{R}^d \)? The absolute value signs now become magnitudes of vectors. Hence one might make the following conjecture:

**Conjecture 1.2** If \( V = \{v_1, \ldots, v_n\} \subseteq \mathbb{R}^2 \), \((\forall i)|v_i| \leq 1\), and \( \sum_{i=1}^{n} v_i = 0 \) then there exists a reordering \( w_1, \ldots, w_n \) of \( V \) such that \((\forall k)|\sum_{i=1}^{k} w_i| \leq 1\).

Alas this is not true. Take the 3 third roots of unity. The sum of any 2 has magnitude bigger than 1. Hence we can never get \(|w_1 + w_2| \leq 1\). We leave it as an exercise to find, for all \( n \), a counterexample.

What if we don’t insist on 1 as an upper bound? What if we want to go to higher dimensions? The following are true:

1. If \( V = \{v_1, \ldots, v_n\} \subseteq \mathbb{R}^2 \), \((\forall i)|v_i| \leq 1\), and \( \sum_{i=1}^{n} v_i = 0 \) then there exists a reordering \( w_1, \ldots, w_n \) of \( V \) such that \((\forall k)|\sum_{i=1}^{k} w_i| \leq \sqrt{5} \approx 2.24\).

2. If \( V = \{v_1, \ldots, v_n\} \subseteq \mathbb{R}^d \), \((\forall i)|v_i| \leq 1\), and \( \sum_{i=1}^{n} v_i = 0 \) then there exists a reordering \( w_1, \ldots, w_n \) of \( V \) such that \((\forall k)|\sum_{i=1}^{k} w_i| \leq \sqrt{(4^d - 1)/3}\).

3. If \( V = \{v_1, \ldots, v_n\} \subseteq \mathbb{R}^2 \), \((\forall i)|v_i| \leq 1\), and \( \sum_{i=1}^{n} v_i = 0 \) then there exists a reordering \( w_1, \ldots, w_n \) of \( V \) such that \((\forall k)|\sum_{i=1}^{k} w_i| \leq 2\).
4. If $V = \{v_1, \ldots, v_n\} \in \mathbb{R}^2$, $(\forall i)[|v_i| \leq 1]$, and $\sum_{i=1}^{n} v_i = 0$ then there exists a reordering $w_1, \ldots, w_n$ of $V$ such that $(\forall k)[\sum_{i=1}^{k} w_i \leq d]$.

We will prove items 1 and 3. Item 2 has a similar proof as item 1, and Item 4 has a similar proof as item 3.

2 The $d = 1$ Case

To prove the $d = 1$ case we first prove a lemma which will be helpful in the $d = 2$ case.

**Def 2.1** Let $A = (a_1, \ldots, a_m)$ and $B = (b_1, \ldots, b_L)$ An *interleaving of $A$ and $B$* is an ordering on the elements $\{a_1, \ldots, a_m, b_1, \ldots, b_L\}$ such that if $i < j$ then $a_i$ will proceed $a_j$ and $b_i$ will proceed $b_j$. As an example, if $m = 3$ and $L = 4$ then $(a_1, b_1, b_2, b_3, a_2, a_3, b_4)$ is an interleaving of $A$ and $B$.

**Lemma 2.2** If $A = (a_1, \ldots, a_m) \in [0, 1]^m$, $B = (b_1, \ldots, b_L) \in [-1, 0]^L$, and $\sum_{x \in A \cup B} x = 0$, then there exists an interleaving of $A$ and $B$, $(w_1, \ldots, w_{m+L})$ such that $(\forall k \leq m+L)[\sum_{i=1}^{k} w_i] \leq 1$.

**Proof:** We define the reorder inductively.

Let

$$w_1 = a_1.$$ 

Assume that $w_1, \ldots, w_p$ are defined, $p < m + L$, and $(\forall k \leq p)[\sum_{i=1}^{k} w_i] \leq 1$.

1. **Case 1:** $\sum_{i=1}^{p} w_i < 0$. There must exist an element of $A$ to draw from since otherwise $\sum_{x \in A \cup B} x < 0$. Let $w_{p+1}$ be the next element of $A$. Clearly $|\sum_{i=1}^{p+1} w_i| \leq 1$.

2. **Case 2:** $\sum_{i=1}^{p} w_i > 0$. There must exist an element of $B$ to draw from since otherwise $\sum_{x \in A \cup B} x > 0$. Let $w_{p+1}$ be the next element of $B$. Clearly $|\sum_{i=1}^{p+1} w_i| \leq 1$. 
3. **Case 3:** \( \sum_{i=1}^{p} w_i = 0 \). If there is an element of \( A \) available to take let \( w_{p+1} \) be the next element of \( A \). If not then if there is an element of \( B \) available. Clearly \[ \left| \sum_{i=1}^{p+1} w_i \right| \leq 1. \]

**Theorem 2.3** If \( V = \{v_1, \ldots, v_n\} \subseteq \mathbb{R}^1 \), \((\forall i) [\left| v_i \right| \leq 1]\), and \( \sum_{i=1}^{n} v_i = 0 \) then there exists a reordering \( w_1, \ldots, w_n \) of \( V \) such that \((\forall k)[\left| \sum_{i=1}^{k} w_i \right| \leq 1]\).

**Proof:** Let \( A \) be the nonnegative elements in \( V \). Let \( B \) be the negative elements in \( V \). Apply Lemma 2.2.

**3 The \( d = 2 \) Case, First Proof**

**Theorem 3.1** If \( V = \{v_1, \ldots, v_n\} \subseteq \mathbb{R}^2 \), \((\forall i) [\left| v_i \right| \leq 1]\), and \( \sum_{i=1}^{n} v_i = 0 \) then there exists a reordering \( w_1, \ldots, w_n \) of \( V \) such that \((\forall k)[\left| \sum_{i=1}^{k} w_i \right| \leq \sqrt{5}]\).

**Proof:**

Let \( U \subseteq V \) be such that \( \left| \sum_{u \in U} u \right| \) is maximized. Let \( u^* = \sum_{u \in U} u \). We can assume that \( u^* \) is of the form \((0, a)\), so it points straight up.

The following facts are easily verified:

1. Every vector in \( V \) that is above or on the \( x \)-axis is in \( V \). (If not then add that vector to \( U \) to form a \( U' \) with \( \left| \sum_{u \in U'} u \right| < \left| \sum_{u \in U} u \right| \).)

2. The sum of the \( x \)-coordinates of the vectors in \( U \) is zero (since \( u^* = (0, a) \)).

3. Every vector in \( V \) that is below the \( x \)-axis is in \( \overline{U} \). (If not then remove that vector from \( U \) to form a \( U' \) with \( \left| \sum_{u \in U'} u \right| < \left| \sum_{u \in U} u \right| \).)

4. The sum of the \( x \)-coordinates of the vectors in \( \overline{U} \) is zero (Since \( \sum_{u \in U} u + \sum_{u \in \overline{U}} u = 0 \) and \( \sum_{u \in \overline{U}} = 0 \)).
Let $U = \{(a_1, b_1), \ldots, (a_m, b_m)\}$ and $\bar{U} = \{(c_1, d_1), \ldots, (c_L, d_L)\}$. By items 2 and 4 we have $\sum_{i=1}^m a_i = 0$ and $\sum_{i=1}^L c_i = 0$. Apply Theorem 2.3 to both $\{a_1, \ldots, a_m\}$ and $\{c_1, \ldots, c_L\}$. By renumbering we can assume that

- $(\forall k \leq m)[|\sum_{i=1}^k a_i| \leq 1]$, and
- $(\forall k \leq L)[|\sum_{i=1}^k c_i| \leq 1]

By items 1 and 3 $(\forall i \leq m)[b_i \geq 0]$ and $(\forall i \leq L)[d_i \leq 0]$. Since $\sum_{i=1}^n v_i = 0$ we know that $\sum_{i=1}^m b_i + \sum_{i=1}^L d_i = 0$. By Lemma 2.2 there is an interlacing $(b_1, \ldots, b_m)$ and $(d_1, \ldots, d_L)$ so that any initial partial sum has absolute value $\leq 1$.

We use this ordering. Let it be

$$(x_1, y_1), \ldots, (x_n, y_n).$$

$$\sum_{i=1}^k (x_i, y_i) = (\sum_{i=1}^k x_i, \sum_{i=1}^k y_i).$$

Since $|\sum_{i=1}^k x_i|$ is the sum of a partial initial sums of the $a_i$’s and of the $b_i$’s by the triangle inequality this quantity is $\leq 2$.

Since we used Lemma 2.2 $|\sum_{i=1}^k y_i| \leq 1$.

Note that in the above few lines we used $\cdot \cdot \cdot$ to mean absolute value. We now use it to mean distance in $\mathbb{R}^2$.

$$\left|\sum_{i=1}^k (x_i, y_i)\right| = \left|\sum_{i=1}^k x_i, \sum_{i=1}^k y_i\right| = \sqrt{(\sum_{i=1}^k x_i)^2 + (\sum_{i=1}^k y_i)^2} \leq \sqrt{2^2 + 1^2} = \sqrt{5}.$$
Theorem 3.2 Let $d \geq 1$. If $V = \{v_1, \ldots, v_n\} \subseteq \mathbb{R}^d$, $(\forall i)[|v_i| \leq 1]$, and $\sum_{i=1}^{n} v_i = 0$ then there exists a reordering $w_1, \ldots, w_n$ of $V$ such that $(\forall k)[\sum_{i=1}^{k} w_i \leq \sqrt{\frac{(d-1)^2}{3}}]$.

4 The $d = 2$ Case, Second Proof

Theorem 4.1 If $V = \{v_1, \ldots, v_n\} \subseteq \mathbb{R}^2$, $(\forall i)[|v_i| \leq 1]$, and $\sum_{i=1}^{n} v_i = 0$ then there exists a reordering $w_1, \ldots, w_n$ of $V$ such that $(\forall k)[\sum_{i=1}^{k} w_i \leq 2]$.

Proof: Rewrite $\sum_{i=1}^{n} v_i = 0$ as

$$\sum_{i=1}^{n} \left( \frac{n-2}{n} \right) v_i = 0$$

and note that

$$\sum_{i=1}^{n} \frac{n-2}{n} = n - 2.$$

We restate this in a less informative way in order to generalize it:

$\exists \alpha_1, \ldots, \alpha_n$ such that

- $(\forall i)[0 \leq \alpha_i \leq 1]$.
- $\sum_{i=1}^{n} \alpha_i v_i = 0$
- $\sum_{i=1}^{n} \alpha_i = n - 2$

We want to derive re-order the $v_i$’s (but we still call them $v_1, \ldots, v_n$), remove $v_n$ from the list, and have the following:

$\exists \beta_1, \ldots, \beta_n$ such that

- $(\forall i)[0 \leq \beta_i \leq 1]$.
- $\sum_{i=1}^{n-1} \beta_i v_i = 0$
- $\sum_{i=1}^{n-1} n\beta_i = n - 3$
We first look at $n$-tuples of $\beta_i$’s and then see if we can make one of them 0. Let

$$TEMP = \{ x \in [0, 1]^n : \sum_{i=1}^n \beta_i v_i = 0 \land \sum_{i=1}^n \beta_i = n - 3 \}. \}$$

Note that

- $(\frac{n-3}{n-2} \alpha_1, \ldots, \frac{n-3}{n-2} \alpha_n) \in TEMP$. In particular $TEMP \neq \emptyset$.
- $TEMP$ is a convex polytope that is a subset of $[0, 1]^n$.
- There is an $n \times 3$ matrix $A$ and a vector $b \in \mathbb{R}^3$ such that

$$TEMP = \{ (\beta_1, \ldots, \beta_n) \in [0, 1]^n : Ax \leq b \}. \}$$

We need a Lemma

**Lemma 4.2** Let $A$ be an $n \times e$ matrix and $b \in \mathbb{R}^e$. Let

$$B = \{ x \in [0, 1]^n : Ax = b \}. \}$$

If $B$ is nonempty then there exists a point in $B$ with $\geq n - e$ of the variables in $\{0, 1\}$

**Proof sketch:** Take an extreme point of $B$. 

$TEMP$ satisfies the condition of $B$ in Lemma 4.2 with $e = 3$. Hence there is a point in $TEMP$ with $\geq n - 3$ coordinates in $\{0, 1\}$. One of the equations is

$$\beta_1 + \cdots + \beta_n = n - 3.$$

Hence there must be an $i$ with $\beta_i = 0$. Reorder to make that $\beta_n$. We are done.

We can keep doing this to obtain, for all $1 \leq k \leq n$, there exists $\beta_1, \ldots, \beta_i$ with
• \( \sum_{i=1}^{k} \beta_i v_i = 0 \)

• \( \sum_{i=1}^{k} \beta_i = k - 2 \)

This gives us a reordering of \( v_1, \ldots, v_n \) (though we still call it \( v_1, \ldots, v_n \)). We are concerned with \(| \sum_{i=1}^{k} v_i |\).

First note that

\[
\sum_{i=1}^{k} v_i = \sum_{i=1}^{k} v_i - \sum_{i=1}^{k} \beta_i v_i = \sum_{i=1}^{k} (1 - \beta_i) v_i.
\]

Hence

\[
| \sum_{i=1}^{k} v_i | = | \sum_{i=1}^{k} (1 - \beta_i) v_i | \leq \sum_{i=1}^{k} |1 - \beta_i| | v_i | \leq \sum_{i=1}^{k} |1 - \beta_i | = \sum_{i=1}^{k} 1 - \beta_i = k - \sum_{i=1}^{k} \beta_i = k - (k - 2) = 2.
\]

Using the same technique one can show the following:

**Theorem 4.3** Let \( d \geq 1 \). If \( V = \{ v_1, \ldots, v_n \} \subseteq \mathbb{R}^d \), \( (\forall i) | v_i | \leq 1 \), and \( \sum_{i=1}^{n} v_i = 0 \) then there exists a reordering \( w_1, \ldots, w_n \) of \( V \) such that \( (\forall k) | \sum_{i=1}^{k} w_i | \leq d \).

**References**