Deriving $\sum_{i=1}^{n} i^k$ Formula

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(This is well known but I do not know a reference or an original reference. If you know one then please email it to me.)

The identity $\sum_{i=1}^{n} i = n(n+1)/2$ has a well known deriviation. We present a deriviation of any $\sum_{i=1}^{n} i^k$ if you know the prior sums.

For $1 \le j \le k-1$ let $S_j = \sum_{i=1}^{n} i^j$.

Note the following

$$\sum_{i=1}^{n} (i+1)^{k+1} - i^{k+1} = (2^{k+1} - 1^{k+1}) + \cdots + ((n+1)^{k+1} - n^{k+1}) = (n+1)^{k+1} - 1.$$

We derive a different expression for this sum and will then equate the two.

$$\begin{aligned}
\sum_{i=1}^{n} (i+1)^{k+1} - i^{k+1} &= \sum_{i=1}^{n} \left(\sum_{j=0}^{k+1} \binom{k+1}{j} i^j\right) - i^{k+1} \\
&= \sum_{i=1}^{n} \sum_{j=0}^{k} \binom{k+1}{j} i^j \\
&= \sum_{j=0}^{k} \binom{k+1}{j} \sum_{i=1}^{n} i^j \\
&= \sum_{j=0}^{k} \binom{k+1}{j} S_j \\
&= \binom{k+1}{k} S_k + \sum_{j=0}^{k-1} \binom{k+1}{j} S_j \\
&= (k+1) S_k + \sum_{j=0}^{k-1} \binom{k+1}{j} S_j
\end{aligned}$$

Hence

$$(k+1)S_k + \sum_{j=0}^{k-1} \binom{k+1}{j} S_j = (n+1)^{k+1} - 1.$$

$$S_k = \frac{1}{k+1}\left((n+1)^{k+1} - 1 - \sum_{j=0}^{k-1} \binom{k+1}{j} S_j \right).$$