A Time Line of Who-Proved-What for Upper bounds on VDW Numbers

An Excerpt from the VDW book being worked on by
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Def 0.0.1 If $A \subseteq \mathbb{N}$, then the *upper density of* A is $\limsup_{n\to\infty} \frac{|A\cap[n]|}{n}$.

We give a history of how better bounds on $W(k,c)$ were found. We also include some side roads.

1. In 1927 van der Waerden proves VDW. His proof yields bounds on $W(k,c)$ that are not primitive recursive.

2. In 1936 Erdős and Turan [?] conjectured that every set of positive upper density has a 3-AP. A proof of this would yield a proof of VDW$(3,c)$ that is very different from the original proof (and from Shelah's proof). They have often been credited with conjecturing that every set of positive upper density has a k-AP with this paper as the reference; however, Soifer [?] gives compelling evidence that the conjecture for k-AP was made by Erdős in the 1957 (next item).

3. In 1953 Roth [?] (see also [?]) proved that, for every $\delta > 0$, for every $N \ge 2^{2^{O(\delta^{-1})}}$, for every $A \subseteq [N]$ with $|A| \ge \delta N$, A has a 3-AP. The proof used Fourier Analysis. This result *did* lead to better bounds on $W(3,c)$, namely $W(3,c) \le 2^{2^{O(c)}}$.

4. In 1957 Erdős [?] conjectured that every set of positive upper density has a k-AP. A proof of this would yield a proof of VDW that is very different from the original proof (and from Shelah's proof). Such a proof might lead to better bounds on $W(k,c)$. We will call this *The Conjecture.*

5. In 1974 Szemerédi [?] proved the $k=4$ case of The Conjecture with a purely combinatorial proof. This result *did not* lead to better bounds on $W(4,c)$. Even though it is purely combinatorial, it is rather difficult.

6. In 1975 Szemerédi [?] proved The Conjecture with a purely combinatorial proof. His result *did not* lead to better bounds on $W(k,c)$ because the proof used VDW. Even though it is purely combinatorial, it is rather difficult. In order to prove this he first proved *Szemerédi's Regularity Lemma* which has been very useful in a variety of fields [?, ?, ?].

7. In 1977 Fürstenberg [**?**] proved The Conjecture with ergodic methods. His proof *did not appear* to lead to *any* bounds on $W(k, c)$ since it was nonconstructive. Avigad and Towsner [**?**] (see also [**?**, **?**, **?**, **?**]) have shown that, in principle, one can extract bounds from the proof; however, these bounds are no better than the classic bounds and may be worse.

8. In 1988 Shelah [**?**] obtained a new proof of VDW that yielded primitive recursive bounds on $W(k, c)$. The bounds are still quite large and cannot be written down. The proof is purely combinatorial and does not use any of the techniques related to The Conjecture.

9. In 1996 Bergelson and Leibman [**?**] used ergodic techniques to prove the following generalization of The Conjecture:

   Let $p_1, \ldots, p_k \in \mathbb{Z}[x]$ such that $(\forall i)[p_i(0) = 0]$. If A is a set of positive upper density then there exists $a, d \in \mathbb{N}$ such that $a, a + p_1(d), a + p_2(d), \ldots, a + p_k(d) \in A$.

   An easy corollary is POLYVDW which we restate here:

   For any polynomials $p_1(x), \ldots, p_k(x) \in \mathbb{Z}[x]$ such that $(\forall i)[p_i(0) = 0]$, for any natural number c, there exists $W = W(p_1, \ldots, p_k; c)$ such that, for any c-coloring $\chi{:}[W] \to [c]$ there exists $a, d \in \mathbb{N}$ such that $\chi(a) = \chi(a + p_1(d)) = \chi(a + p_2(d)) = \cdots = \chi(a + p_k(d))$.

   Their proof *did not appear* to lead to *any* bounds on W since it was nonconstructive. Towsner [**?**] showed that, in principle, one can extract bounds from the proof; however, these bounds are no better than the classic bounds and may be worse.

10. Bourgain [**?**] showed that $A \subseteq [n]$ and $|A| \geq \Omega\left(n\sqrt{\frac{\log\log n}{\log n}}\right)$ then A has a 3-AP. The proof is rather difficult and not purely combinatorial. This can be used to obtain a better bound on $W(3, c)$ then Roth had.

11. In 2000 Walters [**?**] obtained a proof of POLYVDW that yielded bounds $W(p_1, \ldots, p_k; c)$. these bounds were not primitive recursive.

12. In 2001 Gowers [**?**, **?**] proved The Conjecture using Fourier methods. His proof *did* yield better bounds on $W(k, c)$. In particular he obtains

$$W(k,c) \le 2^{2^{c^{2^{2^{k+9}}}}}$$

13. In 2002 Shelah [?] obtained a proof of POLYVDW that yielded primitive recursive bounds on $W(p_1,\ldots,p_k;c)$.

14. In 2006 Graham and Solymosi [?] obtained a purely combinatorial proof that $W(3,c) \le 2^{2^{2^{O(c)}}}$.