Exact Fair Division

1 Introduction

Whenever we say something like *Alice has a piece worth 1/2* we mean worth 1/2 TO HER.

Lets say we want Alice and Bob to split a cake so that each gets EXACTLY 1/2. Is this possible? YES if we allow a Moving Knife Protocol and randomization.

2 Moving Knife Protocol

Def 2.1 Let n be the number of players. An *exact division* is such that each player gets exactly $\frac{1}{n}$.

Theorem 2.2 *There is a MK protocol with randomization for 2-player exact division.*

Proof:

1. Alice holds TWO knives over the cake, one on the left edge. (The other such that the cake between the two knives is 1/2. Note that she must think that inside=outside=1/2.)

2. Alice moves both knives across the cake. (Such that what is in the middle is always 1/2.)

3. Bob yells STOP. (When he thinks inside=outside=1/2.)

4. A coin is flipped. If it comes up HEADS then Bob gets the inside and Alice gets the outside. If it comes up TAILS then Alice gets the inside and Bob gets the outside.

We leave it to the reader to show that (1) if either one cheats they may end up with $< 1/2$, and (2) if neither one cheats then they both end up with exactly 1/2.

We also must show that there will be a point where Bob thinks inside=outside=1/2. There are three cases.

Case 1: In the beginning Bob thinks that what is inside the knives is $< 1/2$. Then he must think what is outside the knives if $> 1/2$. Note that when Alice finishes she has the knives surrounding what was initially the outside part. Hence (a) When Alice begins Bob thinks that between-the-knifes is $< 1/2$, (b) When Alice finishes Bob thinks that between-the-knifes is $> 1/2$. Hence there must be a point in between where he thinks it is $= 1/2$.

Case 2: In the beginning Bob thinks that what is inside the knives is $= 1/2$. Then he yells immediately.

Case 3: In the beginning Bob thinks that what is inside the knives is $> 1/2$. Similar to case 1.

▮

We now extend to 3 players.

Here is our attempt:

Theorem 2.3 *There is a MK protocol with randomization for 3-player exact division.*

Proof:

1. Alice and Bob do the MK protocol from Theorem 2.2. Note that Alice and Bob each have exactly 1/2.

2. Alice and Carol split Alice's piece such that Alice gets exactly 2/3 of it, and Bob gets 1/3 of it. WAIT- WE DO NOT KNOW HOW TO DO THAT. WE QUIT NOW BUT WILL COME BACK TO IT

▮

Def 2.4 Let $0 < \alpha < \beta < 1$ and $\alpha + \beta = 1$. Let $0 < \epsilon < 1$. An (α, β) exact protocol is a protocol such that, at the end, Alice has α of the cake and Bob has β of the cake.

Theorem 2.5 *There is a MK protocol with randomization for* $(2/3, 1/3)$ *exact division.*

Proof:

1. Alice holds TWO knives over the cake, one on the left edge. (The other such that the cake between the two knives is 1/3. Note that she must think that inside=1/3 and outside=2/2.)

2. Alice moves both knives across the cake. (Such that what is in the middle is always 1/3.)

3. Bob yells STOP. (When he thinks inside=1/3.)

4. A coin is flipped. If it comes up HEADS then Bob gets the inside part (1/3) and Alice gets the outside part (2/3) and we are done. If it is TAILS then Alice gets the inside part (1/3). They now do the 2-player exact protocol on the outside. Hence in the end Alice gets $1/3 + (1/2)(2/3)$ and Bob gets $(1/2)(2/3)$.

We leave it to the reader to show that (1) If Alice cheats then she may get $< 2/3$, (2) If Bob cheats then he may get $< 1/3$, and (3) if neither one cheats then Alice gets 2/3 and Bob gets 1/3.

We also must show that there will be a point where Bob thinks inside=1/3. We leave this to the reader.

∎

We can now proof our original theorem.

Theorem 2.6 *There is a MK protocol with randomization for 3-player exact division.*

Proof:

1. Alice and Bob do the MK protocol from Theorem 2.2. Note that Alice and Bob each have exactly 1/2.

2. Alice and Carol do the $(2/3, 1/3)$ exact division on Alice's piece.

3. Bob and Carol do the $(2/3, 1/3)$ exact division on Bob's piece.

Originally Alice has 1/2 and Bob has 1/2. After Alice and Carol finish Alice has $(1/2)(2/3) = 1/3$ After Bob and Carol finish Bob has $(1/2)(2/3) = 1/3$.

What about Carol. Let A be how much Carol values Alice's piece. Let B be how much Carol values Bob's piece. We know $A + B = 1$. Carol gets

$$(1/3)A + (1/3)B = (1/3)(A + B) = 1/3.$$

■

Note 2.7 It is an exercise to show that this can be extended to more n players where each one gets exact $1/n$. You will need a theorem about (α, β, γ)-exact division.

3 Discrete Protocols for Near-Exact Division for Two People

Is there a discrete protocol for exact-division? Alas no- one can show that there is not (we omit the proof). So we try for the next best thing.

Def 3.1 Let n be the number of players. Let $0 < \epsilon < 1$. A division is *ϵ-near exact* if in the end everyone gets a piece of value in the interval $[\frac{1}{n} - \epsilon, \frac{1}{n} + \epsilon]$.

Theorem 3.2 *For all L there is a discrete 2-player protocol for $\frac{1}{L}$-near exact division.*

Proof:

1. Alice cuts the pie into $2L$ pieces. (Evenly, so each piece is $\frac{1}{2L}$.) (Later I will point out why we use $\frac{1}{2L}$.)

2. Bob takes some pieces and cuts them further (Any piece that he thinks is $> \frac{1}{2L}$ he cuts into pieces so that each piece is $\leq \frac{1}{2L}$.)

3. Alice and Bob write down privately and simultaneously what each piece is worth to them. They reveal these numbers.

4. Let the pieces be $p_1, \ldots, p_m$. Let value$_A(p_i)$ be how much A values piece p_i. Let value$_B(p_i)$ be how much A values piece p_i. (Note that if neither cheated then, for all i, value$_A(p_i)$, value$_B(p_i) \le \frac{1}{2:wL}$.)

5. Alice and Bob reorder the pieces.

   - Let $q_1 = p_1$.
   - Assume that $q_1, \ldots, q_k$ are already defined.
     - If
       $$\sum_{i=1}^{k} \text{value}_A(q_i) < \sum_{i=1}^{k} \text{value}_B(q_i)$$
       then Alice and Bob find a piece p not already used such that value$_B(p) >$ value$_A(p)$. Let $q_{k+1} = p$.
     - If
       $$\sum_{i=1}^{k} \text{value}_A(q_i) > \sum_{i=1}^{k} \text{value}_B(q_i)$$
       then Alice and Bob find a piece p not already used such that value$_B(p) <$ value$_A(p)$. Let $q_{k+1} = p$.
     - If
       $$\sum_{i=1}^{k} \text{value}_A(q_i) = \sum_{i=1}^{k} \text{value}_B(q_i)$$
       then Alice and Bob find a piece p not already used such that value$_B(p) \le$ value$_A(p)$. Let $q_{k+1} = p$. (They could have done value$_A(p) \ge$ value$_B(p)$ it would not have mattered.)

   NOTE- it is an exercise to show that you can always find the piece p that you need.

6. We now have $q_1, \ldots, q_m$. Let k be the least number such that
   $$\sum_{i=1}^{k} \text{value}_A(q_i) \ge \frac{1}{2}.$$
   Alice gets $q_1, \ldots, q_k$. Bob gets $q_{k+1}, \ldots, q_m$.

We show that if Alice and Bob do not cheat then they each get within $\frac{1}{L}$ of 1/2. (We leave it as an exercise that if either one cheats they may get LESS than this.)

Notation 3.3 Let $1 \le j \le m$, Let $S_A^j = \sum_{i=1}^{j} \text{value}_A(q_i)$. Let $S_B^j = \sum_{i=1}^{j} \text{value}_B(q_i)$.

KEY FACT: For all j,

$$|S_A^j - S_B^j| \le \frac{1}{2L}.$$

Proof of KEY FACT: We first show that this holds at $j = 1$. All this means is that $|\text{value}_A(q_1) - \text{value}_B(q_1)| \le \frac{1}{2L}$. Since each of these values if between 0 and $\frac{1}{2L}$ the difference is at most $\frac{1}{2L}$

Intuitively, whenever one of the sums is behind we define q so that it catches up. This is why the difference of the sums can never get to large. We show this formally by a technique known as induction. We show that if the statement holds for j then it holds for $j+1$. This is enough to establish that it holds for all j.

Assume

$$|S_A^j - S_B^j| \le \frac{1}{2L}.$$

Assume that $S_A^j > S_B^j$, hence $|S_A^j - S_B^j| = S_A^j - S_B^j$. The other cases are similar. Alice and Bob then find a p such that $\text{value}_A(p) < \text{value}_B(p)$ and let $q_{j+1} = p$. Note that

$S_A^{j+1} = S_A^j + \text{value}_A(p)$

$S_B^{j+1} = S_B^j + \text{value}_B(p)$

Also note that

$$0 \le S_A^j - S_B^j \le \frac{1}{2L}$$

$$-\frac{1}{2L} \le \text{value}_A(p) - \text{value}_B(p) \le 0.$$

Add these two together to get

$$-\frac{1}{2L} \le S_A^j + \text{value}_A(p) - S_B^j - \text{value}_B(p) \le \frac{1}{2L}$$

$$-\frac{1}{2L} \le (S_A^j + \text{value}_A(p)) - (S_B^j + \text{value}_B(p)) \le \frac{1}{2L}.$$

$$-\frac{1}{2L} \le S_A^{j+1} - S_B^{j+1} \le \frac{1}{2L}.$$

Hence

$$|S_A^{j+1} - S_B^{j+1}| \le \frac{1}{2L}.$$

End of Proof of KEY FACT

We now show that Alice is within $\frac{1}{L}$ of 1/2 (which will not use KEY FACT) and that Bob is within $\frac{1}{L}$ of 1/2 (which will use KEY FACT).

ALICE: Alice gets S_A^k where $S_A^{k-1} < \frac{1}{2}$ and $S_A^k \ge \frac{1}{2}$. Note that

$$S_A^k = S_A^{k-1} + \text{value}_A(q_k)$$

Since $S_A^{k-1} < \frac{1}{2}$ and $\text{value}_A(q_k) \le \frac{1}{2L}$

$$S_A^k = S_A^{k-1} + \text{value}_A(q_k) \le \frac{1}{2} + \frac{1}{2L}.$$

Combine this with $S_A^k \ge \frac{1}{2}$ to obtain

$$\frac{1}{2} \le S_A^k \le \frac{1}{2} + \frac{1}{2L}.$$

This is stronger than we need which is just that Alice gets within $\frac{1}{L}$ of $\frac{1}{2}$.

BOB: Bob gets $q_{k+1}, \ldots, q_m$ which he values at $1 - S_B^k$. We will show that (1) Alice values $q_{k+1}, \ldots, q_m$ at close to $\frac{1}{2}$, (2) Bob and Alice's values are close together, hence (3) Bob values $q_{k+1}, \ldots, q_m$ at close to $\frac{1}{2}$.

Recall that

$$\frac{1}{2} \le S_A^k \le \frac{1}{2} + \frac{1}{2L}.$$

Also recall from KEY FACT that

$$-\frac{1}{2L} \le S_B^k - S_A^k \le \frac{1}{2L}$$

Adding these two equations we get

$$\frac{1}{2} - \frac{1}{2L} \le S_B^k \le \frac{1}{2} + \frac{1}{2L} + \frac{1}{2L}$$

(NOTE- this is why we picked $\frac{1}{2L}$ in the first place— since we knew that we would end up within twice that value which is $\frac{1}{L}$.)

$$\frac{1}{2} - \frac{1}{2L} \le S_B^k \le \frac{1}{2} + \frac{1}{L}$$

We are really interested in $1 - S_B^k$. So lets negate all sides and add 1.

First we negate all sides and switch the inequality:

$$-\frac{1}{2} + \frac{1}{2L} \ge -S_B^k \ge -\frac{1}{2} - \frac{1}{L}$$

Now we rewrite the inequality:

$$-\frac{1}{2} - \frac{1}{L} \le -S_B^k \le -\frac{1}{2} + \frac{1}{2L}$$

Now we add 1 to all sides:

$$1 - \frac{1}{2} - \frac{1}{L} \le 1 - S_B^k \le 1 - \frac{1}{2} + \frac{1}{2L}$$

$$\frac{1}{2} - \frac{1}{L} \le 1 - S_B^k \le \frac{1}{2} + \frac{1}{2L}.$$

So Bob gets within $\frac{1}{L}$ of $\frac{1}{2}$. ∎

4 Discrete Protocols for Near-Exact Division for Three People

Theorem 4.1 *For all L there is a discrete 3-player protocol for $\frac{1}{L}$-near exact division.*

Proof:

1. Alice and Bob do the $\frac{1}{M}$-near exact protocol. (We will pick M later.)

2. Alice and Carol WANT to do a protocol so that Alice gets approx 2/3 of her piece and Carol gets approx 1/3 of the piece. OH- WE DO NOT KNOW HOW TO DO THAT! WE CANNOT USE THE PRIOR ALGORITHM. WE NEED SOMETHING ELSE.

WE ARE GIVING UP FOR NOW. ▮

What the above shows is that we need a 2-person protocol for unfair near-exact division.

Def 4.2 Let $0 < \alpha < \beta < 1$ and $\alpha + \beta = 1$. Let $0 < \epsilon < 1$. An (α, β)-ϵ-near exact protocol is a protocol such that, at the end, Alice has within ϵ of α of the cake and Bob has within ϵ of β of the cake.

Theorem 4.3 *For all L there is a discrete 2-player protocol for $(2/3, 1/3)$-$\frac{1}{L}$-near exact division.*

Proof:

1. The first five steps of the protocol are the same as for the protocol in Theorem 3.2
2. At this point there are m pieces $q_1, \ldots, q_m$ such that
   - $\text{value}_A(q_i) \le \frac{1}{2L}$.
   - $\text{value}_B(q_i) \le \frac{1}{2L}$.
   - **KEY FACT:** For all j,
   $$|S_A^j - S_B^j| \le \frac{1}{2L}.$$
3. Let k be the least number such that $S_A^k \ge \frac{2}{3}$. Alice gets $q_1, \ldots, q_k$. Bob gets $q_{k+1}, \ldots, q_m$.

We leave it to the reader to prove that this works.

▮

We can now do the proof we tried to do before.

Theorem 4.4 *For all L there is a discrete 3-player protocol for $\frac{1}{L}$-near exact division.*

Proof:

M is a parameter that will depend on L. We will later pick a value of M that makes the proof work.

1. Alice and Bob do the $\frac{1}{M}$-near exact protocol.
2. Alice and Carol do the $(\frac{2}{3}, \frac{1}{3})$-$\frac{1}{M}$-near exact protocol.
3. Bob and Carol do the $(\frac{2}{3}, \frac{1}{3})$-$\frac{1}{M}$-near exact protocol.

After step 1
Alice has V_A where

$$\frac{1}{2} - \frac{1}{M} \le V_A \le \frac{1}{2} + \frac{1}{M}$$

and
Bob has V_B where

$$\frac{1}{2} - \frac{1}{M} \le V_B \le \frac{1}{2} + \frac{1}{M}.$$

After the second step Alice has V'_A where

$$\frac{2V_A}{3} - \frac{1}{M} \le V'_A \le \frac{2V_A}{3} + \frac{1}{M}$$

We obtain a lower bound on V'_A. We start with the lower bound on V_A.

$$\begin{aligned}
V_A &\ge \tfrac{1}{2} - \tfrac{1}{M} \\
\tfrac{2V_A}{3} &\ge \tfrac{2}{3}(\tfrac{1}{2} - \tfrac{1}{M}) \\
\tfrac{2V_A}{3} &\ge \tfrac{1}{3} - \tfrac{2}{3M}) \\
\tfrac{2V_A}{3} - \tfrac{1}{M} &\ge \tfrac{1}{3} - \tfrac{2}{3M} - \tfrac{1}{M} \\
V'_A \ge \tfrac{2V_A}{3} - \tfrac{1}{M} &\ge \tfrac{1}{3} - \tfrac{5}{3M}
\end{aligned}$$

Hence we need $\frac{5M}{3} \le \frac{1}{L}$.
We obtain an upper bound on V'_A. We start with the upper bound on V_A.

$$
\begin{array}{rcl}
V_A & \le & \frac{1}{2} + \frac{1}{M} \\
\frac{2V_A}{3} & \le & \frac{2}{3}(\frac{1}{2} + \frac{1}{M}) \\
\frac{2V_A}{3} & \le & \frac{1}{3} + \frac{2}{3M} \\
\frac{2V_A}{3} + \frac{1}{M} & \ge & \frac{2}{3} + \frac{2}{3M} + \frac{1}{M} \\
\frac{2V_A}{3} + \frac{1}{M} & \ge & \frac{1}{3} + \frac{5}{3M} \\
V_A' \le \frac{2V_A}{3} + \frac{1}{M} & \ge & \frac{1}{3} + \frac{5}{3M}
\end{array}
$$

Hence we need $\frac{5M}{3} < \frac{1}{L}$.

So, if $\frac{5}{3M} < \frac{1}{L}$ then Alice will get within $\frac{1}{L}$ of $\frac{1}{3}$. By similar reasoning we also get that Bob will get within $\frac{1}{L}$ of $\frac{1}{3}$.

We now look at Carol. Initially Carol thinks Alice's piece is worth A and Bob's piece is worth B. All we know is that $A + B = 1$.

Carol gets within $\frac{1}{M}$ of $\frac{A}{3}$ from Alice. Carol gets within $\frac{1}{M}$ of $\frac{B}{3}$ from Bob. Hence Carol gets at least

$$\frac{A}{3} - \frac{1}{M} + \frac{B}{3} - \frac{1}{M} = \frac{A+B}{3} - \frac{2}{M} \ge \frac{1}{3} - \frac{2}{M}.$$

Hence we need $\frac{2}{M} \le \frac{1}{L}$.

Similar reasoning can be used to show that if $\frac{2}{M} \le \frac{1}{L}$ then Carol gets at most $\frac{1}{3} + \frac{1}{L}$.

Hence we need just two inequalities:

$$\frac{5}{3M} \le \frac{1}{L}$$

and

$$\frac{2}{M} \le \frac{1}{L}.$$

Take $M = 2L$.

∎