MIDTERM for HONR 209M
SOLUTIONS

1. (0 points) What is your name. Write clearly.

2. (20 points) Consider the following scenario:

item	Alice	Bob	ratio Alice/Bob	ratio Bob/Alice
House	40	12	3.33...	0.3
Car	10	15	0.66...	1.5
Dog	10	3	3.33...	0.3
Pension	10	12	0.833...	1.2
Tool Set	10	16	0.625	1.6
Zebra	20	42	0.4166...	2.4

All items are continous EXCEPT the car. Hence all items, EXCEPT the car, can be split up.

Perform the AW algorithm on this problem Assume that all items are continous. (Recall that we split the item that winner has that has the smallest ratio between the winner's valuation and the non-winners valuation. If there is a tie then take one arbitrarily.) State who gets what items and parts of items. Also state how many points each person gets in the end.

SOLUTION TO PROBLEM 2

The following happen in both parts:

Alice initially gets House and Dog for $40 + 10 = 50$ points.

Bob initially gets Car, Pension, Tool Set, Zebra for $15 + 12 + 16 + 42 = 85$.

So Bob has more.

The item that Bob won that has the smallest ratio of valuations is the Pension.

$$\begin{aligned} 85 - 12z &= 50 + 10z \\ 35 &= 22z \\ z &= 35/22 \end{aligned}$$

This is greater than 1. So Bob gives the entire Pension to Alice.

Now

Alice gets House, Dog, and Pension for a total of $50 + 10 = 60$.

Bob gets Car, Tool Set, and Zebra for a total of $85 - 12 = 73$.

Still need to give Alice more. The next item that has the smallest valuation ration is the Car. BUT the Car is not splittable. The next item is the Tool Set. So Bob will give Alice some of his Tool Set.

$$\begin{aligned} 73 - 16z &= 60 + 10z \\ 13 &= 26z \\ z &= 13/26 = 1/2 \end{aligned}$$

So Bob gives Alice 1/2 of the Tool Set.

Alice has House, Dog, Pension, and 1/2 of the Tool Set. This is worth $40+10+10+(1/2)*10 = 65$.

Bob has Car, Zebra, and 1/2 of the Tool Set. This is worth $15+42+(1/2)*16 = 57+8 = 65$.

3. (20 points). Usually in the AW protocol the goal is to have Alice and Bob get the same number of points in their own view.

   (a) Describe a variant of AW that gives one player twice as many points as the other. Include all things that could happen (e.g., they both like an item the same amount, the value of x then get is > 1.) (HINT- The protocol is the same as AW until you get to the part where you solve for x.)

   (b) Apply your protocol to the following example where you want Alice to get twice as much as Bob.

item	Alice	Bob	ratio Alice/Bob	ratio Bob/Alice
House	40	20	2.00...	0.5
Car	10	20	0.50...	2.0
Dog	15	25	0.60...	1.66...
Pension	15	30	0.50...	2.0
Tool Set	20	5	4.00	0.25

SOLUTION TO PROBLEM 3

We want a protocol so that Alice gets twice as much as Bob.

Protcol

(a) Alice and Bob both write down, for each item, how much it is worth to them. The point totals must add up to 100. (This is identical to the usual AW.)

(b) Alice gets all the items that she values more than Bob. Bob gets all the items that she values more than Alice. Put the tied ones aside for now.

(c) Let A_0 be how much Alice has and B_0 be how much Bob has now. Divide up the items in the tied pile such that, if A_1 is how much Alice has after the tied are distributed, and B_1 is how much Bob has after the tied are distributed, then $|A_1 - 2B_1|$ is minimized.

(d) Let A be how much Alice has and B be how much Bob has. There are two cases.

- If $A < 2B$ then we need to give Alice more. Take the item with the lowest ratio of Bob-likes to Alice-likes. Call that item I. Let I_A be how much Alice likes it and I_B be how much Bob likes it. Solve the equation

$$A + xI_A = 2(B - xI_B)$$

  Use this to give Alice x fraction of I. If $x > 1$ then give Alice the entire item. Let A and B be the new values. If $A = 2B$ then you are done. If $A < 2B$ then repeat with next item with lowest ratio.
- If $A > 2B$ then we need to give Bob more. Take the item with the lowest ratio of Alice-likes to Bob-likes. Call that item I. Let I_A be how much Alice likes it and I_B be how much Bob likes it. Solve the equation

$$A - xI_x = 2(B + xI_B)$$

  Use this to give Bob x fraction of I. If $x > 1$ then give Bob the entire item. Let A and B be the new values. If $A = 2B$ then you are done. If $A > 2B$ then repeat with next item with lowest ratio.

We now apply this to the example. I put an A for Alice and a B for Bob in the table below to indicate who got the item after step 1.

item	Alice	Bob	ratio Alice/Bob	ratio Bob/Alice
House-A	40	20	2.00...	0.5
Car-B	10	20	0.50...	2.0
Dog-B	15	25	0.60...	1.66...
Pension-B	15	30	0.50...	2.0
Tool Set-A	20	5	4.00	0.25

Alice has House, Tool Set, for a total of 60.

Bob has Car, Dog, Pension for a total of 75.

Note that $A < 2B$. So we need Bob to give (part of) either the Car, Dog, or Pension to Alice. Dog has the lowest ratio of Bob-likes to Alice-likes, so we use that

$$\begin{aligned} 60 + 15x &= 2(75 - 25x) \\ 60 + 15x &= 150 - 50x \\ 65x &= 90 \\ x &= 90/65 \end{aligned}$$

Note that $x > 1$ so we give the ENTIRE dog to Alice.

NOW

Alice has House, Tool Set, Dog for a total of 75.

Bob has Car, Pension for a total of 50.

Note that $A < 2B$. So we need Bob to give (part of) Dog or Pension to Alice. Lets do the Pension (it was in problem 2 that we said that Car was not divisible, not this problem. However, the Pension is fine.)

$$\begin{aligned} 75 + 15x &= 2(50 - 30x) \\ 75 + 15x &= 100 - 60x \\ 75x &= 25 \\ x &= 25/75 = 1/3 \end{aligned}$$

So Bob gives Alice 1/3 of the pension.

Alice has House, Tool Set, Dog, and 1/3 of the pension, for a total of $75 + 15/3 = 80$.

Bob has Car and 2/3 of the pension for a total of $20 + (2/3)30 = 40$.

Note that Alice has twice as much as Bob.

Some notes on this:

(a) Some in the class had a different way of picking which item to distribute first. This is fine.

(b) Some said to INITIALLY try to give Alice about twice as much as Bob instead of the *the one who likes it gets it*. This is fine so long as the way to later transfer items from one to the other is as above (or similar).

GOTO NEXT PAGE.

4. (20 point) For this problem we use the Point values of problem 2. Assume Alice and Bob did the Moving Knife Protocol in the order

   house, car, dog, pension, tool set, zebra.

   Assume Alice knows Bob's point values. and they did the Moving Knife Protocol. What should she do to maximize her points? Once she does that:

   (a) Which items does Alice get? How many points are they worth to Alice?

   (b) Which items does Bob get? How many points are they worth to Bob?

   (c) How many points does Alice think Bob got?

   (d) How many points does Bob think Alice got?

   SOLUTION TO PROBLEM 4

   Recall:

item	Alice	Bob	ratio Alice/Bob	ratio Bob/Alice
House	40	12	3.33...	0.3
Car	10	15	0.66...	1.5
Dog	10	3	3.33...	0.3
Pension	10	12	0.833...	1.2
Tool Set	10	16	0.625	1.6
Zebra	20	42	0.4166...	2.4

   Alice KNOWS Bob's preferences. Hence Alice KNOWS that Bob will not stop the knife until right before the Tool Set. Hence Alice will yell STOP right before the Tool Set (or allow Bob to). If she does this then

   (a) Which items does Alice get? How many points are they worth to Alice? Alice gets House, Car, Dog, Pension for a total of 70 points.

   (b) Which items does Bob get? How many points are they worth to Bob? Bob gets Tool Set and Zebra for a total of 58 points.

   (c) How many points does Alice think Bob got? Alice values Tool Set and Zebra as 10+20= 30 points.

   (d) How many points does Bob think Alice got? Bob values House+Car+Dog+Pension as 12+15+3+12 = 42.

5. (24 points) In the Succ Pairs algorithm we assume that it begins with Alice and Bob doing cut-and-choose, with Alice cutting and Bob choosing. We assume that Succ Pairs is used in this problem. For each of the following EITHER give a scenario where it happens OR argue that it CANNOT happen.

(a) All players play honestly and Alice thinks she got MORE than 1/3 and Bob thinks he got MORE THAN 1/3.

(b) All players play honestly and Bob thinks he got MORE than 1/3 and Carol thinks she got MORE THAN 1/3.

(c) Alice plays dishonestly and gets MORE than 1/3.

(d) Carol plays dishonestly and gets MORE than 1/3.

(a) All players play honestly and Alice thinks she got MORE than 1/3 and Bob thinks he got MORE THAN 1/3. **THIS CANNOT HAPPEN.** Alice cuts the cake in 1/2. So when she does Cut and Choose with Bob she has 1/2 in her eyes. Then Alice cuts her 1/2 into thirds. Carol takes one of those thirds. So Alice thinks she has 1/2 - 1/6 = 1/3.

(b) All players play honestly and Bob thinks he got MORE than 1/3 and Carol thinks she got MORE THAN 1/3. **THIS CAN HAPPEN.** Alice cuts the cake in a way that Bob and Carol thinks is 0-1 with Bob getting the 1-part. Then Bob cuts it in thirds in a way that Carol thinks it is cut 0-0-1. Carol gets 1, Bob gets 1-1/3 = 2/3.

(c) Alice plays dishonestly and gets MORE than 1/3. **THIS CAN HAPPEN.** Alice cuts it 1/4-3/4 (perhaps hoping Bob will take 1/4) but Bob takes the 3/4 piece. Alice then cuts the 1/4 in thirds and ends up with 1/4 - 1/12 = 1/6.

(d) Carol plays dishonestly and gets MORE than 1/3. **THIS CANNOT HAPPEN (THOUGH SEE NOTE BELOW)**: Carol just chooses pieces to take. she does so cutting so there is nothing she can do to make her get more cake.

NOTE: Some pointed out that Carol can END UP with more, though not by cheating. This is true, and I gave full credit for it.

6. (16 points) Do a cut-diagram for unequal division using the standard algorithm where the initial ratio is (21:16). You may stop each branch when it gets to a ratio of the form $(a : 1)$ since we know that requires $\lfloor \log_2(a) \rfloor + 1$ cuts in the worst case. How many cuts are needed in the worse case?

SOLUTION TO PROBLEM 6

Will do this in class- hard to do diagrams with my word processor.