Strong Fair Division

Whenever we say something like *Alice has a piece worth 1/2* we mean worth 1/2 TO HER.

Lets say we want Alice and Bob to split a cake so that each one thinks they got MORE THAN 1/2. Is this possible? No- if Alice and Bob have the exact same opinion of the cake then this is not possible. But if there is some piece they differ on then it is possible.

Theorem 0.1 *Assume that there is a cake with a piece already cut out of it. Assume that Alice and Bob disagree on the value of that piece. Then there is a 2-person protocol that produces a division so that Alice has a piece worth MORE THAN* 1/2 *and Bob has a piece worth MORE THAN* 1/2*.*

Proof: Let P be the piece that Alice and Bob differ on. Let Q be the rest of the cake.

Assume Alice thinks P is worth α, Bob thinks P is worth β, and and c, d are such that

$$\alpha < \frac{c}{d} < \beta.$$

Note that:

- Alice thinks P is worth $< c/d$. Hence Alice thinks Q is worth $> 1 - \frac{c}{d} = \frac{d-c}{d}$.
- Bob thinks that Q is worth $> \frac{c}{d}$.

We now present the protocol.

1. Alice cuts Q into $d - c$ pieces. (Evenly, so Alice thinks each mini-piece is worth MORE THAN $\frac{1}{d}$.) Call these pieces $a_1, \ldots, a_{d-c}$. Note that at this point neither Alice nor Bob has any of the cake.
2. Bob cuts Q into d pieces. (Evenly, so Bob thinks each mini-piece is worth MORE THAN $\frac{1}{d}$.) Call these pieces $b_1, \ldots, b_d$. Note that at this point neither Alice nor Bob has any of the cake.
3. Alice picks one of the b_i's. This piece is now given to Bob. Call this piece b_{i_0}. (Alice picks the smallest b_i.) KEY: Since Alice thinks P is worth LESS THAN $\frac{c}{d}$ she must think b_{i_0} is worth LESS THAN $\frac{1}{d}$.)

4. Bob picks one of the a_j's. This piece is now given to Alice. Call this piece a_{j_0}. (Bob picks the smallest a_j.) KEY: Since Bob thinks P is worth LESS THAN $\frac{c}{d}$ she must think a_{j_0} is worth LESS THAN $\frac{1}{d}$.)

5. Alice and Bob do cut and choose on the cake that is left. That is, the cake minus the $\{a_{j_0}, b_{i_0}\}$ that are already allocated.

We show that Alice and Bob each think they have MORE THAN $\frac{1}{2}$. View the cake as the union of $\{a_{j_0}, b_{i_0}\}$ (Abbreviated SET) and EVERYTHINGELSE (abbreviated EE). Note that both Alice and Bob think that SET and EE together are worth 1.

Since Alice and Bob did Cut and Choose on EE Alice thinks she got AT LEAST $\frac{1}{2}$ of EE and Bob thinks he got AT LEAST $\frac{1}{2}$ of EE.

Consider the set $\{a_{j_0}, b_{i_0}\}$

Alice thinks:

$$\text{value}(b_{i_0}) < \frac{1}{d} < \text{value}(a_{j_0}).$$

Alice got a_{j_0}. Hence Alice thinks she got MORE THAN $\frac{1}{2}$ of SET. Lets say she thinks he got $\frac{\text{value}(SET)}{2} + \delta_A$.

Bob thinks.

$$\text{value}(a_{j_0}) < \frac{1}{d} < \text{value}(b_{i_0}).$$

Bob got b_{i_0}. Hence Bob thinks she got MORE THAN $\frac{1}{2}$ of SET. Lets say he thinks he got $\frac{\text{value}(SET)}{2} + \delta_B$.

TOTALS:

Alice thinks she got at least

$$\frac{\text{value}(EE)}{2} + \frac{\text{value}(SET)}{2} + \delta_A = \frac{1}{2}(\text{value}(EE) + \text{value}(SET)) + \delta_A = \frac{1}{2} + \delta_A.$$

Bob thinks she got at least

$$\frac{\text{value}(EE)}{2} + \frac{\text{value}(SET)}{2} + \delta_B = \frac{1}{2}(\text{value}(EE) + \text{value}(SET)) + \delta_A = \frac{1}{2} + \delta_B.$$

Hence they both think they have MORE THAN $\frac{1}{2}$.

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We will use the protocol in Theorem 0.1 to show the following.

Theorem 0.2 *Assume that there is a cake with a piece already cut out of it. Assume that Alice and Bob disagree on the value of that piece. Then there is a 3-person protocol that produces a division so that Alice has a piece worth MORE THAN* $1/3$*, Bob has a piece worth MORE THAN* $1/3$*, and Carol has a piece worth MORE THAN* $1/3$*.*

Proof:

We give the protocol. There will be one parameter in the protocol that we later derive what it should be.

1. Alice and Bob do the protocol from Theorem 0.1.
2. Let L be such that Alice and Bob both have a piece worth $\geq \frac{1}{2} + \frac{1}{L}$
3. Let k be a parameter to be named later. It will depend on L.
4. Alice divides her piece into $3k - 1$ pieces. (Evenly)
5. Carol takes k of the pieces. (The top k pieces)
6. Bob divides his piece into $3k - 1$ pieces. (Evenly)
7. Carol takes k of the pieces. (The top k pieces).

We show that everyone gets MORE THAN 1/3. Actually we will pick a value of k so that this is the case.

Carol: Let A be how much Alice's piece is worth to Carol Let B be how much Bob's piece is worth to Carol. Note that $A + B = 1$. Carol ends up with at least

$$\frac{k}{3k-1}A + \frac{k}{3k-1}B = \frac{k}{3k-1}(A+B) = \frac{k}{3k-1} > 1/3.$$

Note that this works for any value of k.

Alice: Alice ends up with

$$\left(\frac{1}{2} + \frac{1}{L}\right) - \frac{k}{3k-1}\left(\frac{1}{2} + \frac{1}{L}\right)$$

$$= \left(\frac{1}{2} + \frac{1}{L}\right)\left(1 - \frac{k}{3k-1}\right)$$

$$= \frac{2k-1}{3k-1}\left(\frac{1}{2} + \frac{1}{L}\right).$$

$$= \frac{2k-1}{3k-1}\frac{L+1}{2L}.$$

We want this to be $> 1/3$. We will pick k so that this happens.

$$\begin{aligned}
\tfrac{2k-1}{3k-1}\tfrac{L+1}{2L} &\geq \tfrac{1}{3} \\
(6k-3)(L+1) &\geq 2L(3k-1) \\
6kL - 3L + 6k - 3 &\geq 6kL - 2L \\
-3L + 6k - 3 &\geq -2L \\
6k - 3 &\geq L \\
6k &\leq L + 3 \\
k &\geq (L+3)/6
\end{aligned}$$

Take k to be the smallest integer that is $\geq (L+3)/6$. We also call this $\lceil (L+3)/6 \rceil$.

Bob: Similar to Alice.

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