1 Introduction

In high school we are taught the following:

1. \( \cos(\pi/1) = -1 \)
2. \( \cos(\pi/2) = 0 \)
3. \( \cos(\pi/3) = \frac{1}{2} \)
4. \( \cos(\pi/4) = \frac{\sqrt{2}}{2} \)
5. \( \cos(\pi/6) = \frac{\sqrt{3}}{2} \)

Note that \( \cos(\pi/5) \) is missing. In Harold Boas’s paper [1] he shows that

\[
\cos(\pi/5) = \frac{1 + \sqrt{5}}{4}
\]

which is half the golden ratio. That paper is the inspiration for this paper.

Are numbers of the form \( \cos(a\pi/b) \) with \( a, b \in \mathbb{N} \) always algebraic? Yes. This is well known. In this paper we prove the theorem with an eye towards (a) getting explicit polynomials, and (b) seeing what the degree of those polynomials are.

2 Chebyshev Polynomials of the First Kind

Def 2.1 The Chebyshev Polynomials of the first kind are, for all \( n \in \mathbb{N} \),

\[
T_n(x) = \sum_{k=0}^{[n/2]} \binom{n}{2k} (x^2 - 1)^k x^{n-2k} \quad [2]
\]

The following theorem is well known.

Theorem 2.2 For all \( n \in \mathbb{N} \), \( T_n(\cos(\theta)) = \cos(n\theta) \).
3 When is \( \cos \left( \frac{a\pi}{b} \right) = \cos \left( \frac{na\pi}{b} \right) \)?

Lemma 3.1

1. Let \( n \in \mathbb{N} \). For all

\[ \theta \in \left\{ \frac{2k\pi}{n-1} : k \in \mathbb{Z}, n > 1 \right\} \cup \left\{ \frac{2k\pi}{n+1} : k \in \mathbb{Z} \right\} \]

\( \cos(\theta) = \cos(n\theta) \).

2. If \( n \) is odd then the \( n \) roots of \( T_n(x) - x = 0 \) are

\[ \left\{ \cos \left( \frac{2k\pi}{n-1} \right) : 0 \leq k \leq \frac{n-1}{2} \right\} \cup \left\{ \cos \left( \frac{2k\pi}{n+1} \right) : 1 \leq k \leq \frac{n-1}{2} \right\}. \]

(If \( n = 1 \) then just take the second union)

3. If \( n \) is even then the \( n \) roots of \( T_n(x) - x = 0 \) are

\[ \left\{ \cos \left( \frac{2k\pi}{n-1} \right) : 0 \leq k \leq \frac{n-2}{2} \right\} \cup \left\{ \cos \left( \frac{2k\pi}{n+1} \right) : 1 \leq k \leq \frac{n}{2} \right\}. \]

(If \( n = 1 \) then just take the second union)

Proof:

1) For the first union notice that

\[ \cos \left( \frac{2k\pi}{n-1} \right) = \cos \left( \frac{2k\pi}{n-1} + 2k\pi \right) = \cos \left( \frac{n2k\pi}{n-1} \right). \]

For the second union notice that

\[ \cos \left( \frac{2k\pi}{n+1} \right) = \cos \left( -\frac{2k\pi}{n+1} \right) = \cos \left( 2\pi k - \frac{2k\pi}{n+1} \right) = \cos \left( \frac{n2k\pi}{n+1} \right). \]

2) By Theorem 2.2 and Part 1 we have that all of the elements in

\[ X = \left\{ \cos \left( \frac{2k\pi}{n-1} \right) : 0 \leq k \leq \frac{n-1}{2} \right\} \cup \left\{ \cos \left( \frac{2k\pi}{n+1} \right) : 1 \leq k \leq \frac{n-1}{2} \right\} \]

are roots of \( T_n(x) - x = 0 \). By algebra one can see that all of the angles mentioned are distinct and in \([0, \pi]\). Since cosine is injective on \([0, \pi]\), \( X \) has \( n \) different numbers. Since \( T_n(x) - x \) is of degree \( n \), the elements of \( X \) are its \( n \) roots.

3) Similar to the proof of Part 2.
4 A Polynomial for $\cos \left( \frac{k\pi}{m} \right)$: $m$ Odd

Def 4.1 Let $m \in \mathbb{N}$, $m$ odd. Take

$$f_{\text{odd}}(m) = \left| \left\{ k : \left( 1 \leq k \leq \frac{m-1}{2} \right) \land \gcd(k, m) = 1 \right\} \right|.$$  

$f_{\text{odd}}(m)$ is the number of numbers in $\{1, \ldots, \frac{m-1}{2} \}$ that are relatively prime to $m$.

Lemma 4.2 Let $m \in \mathbb{N}$, $m \geq 3$, $m$ odd. Let

$$A_{m,1} = \left\{ \cos \left( \frac{2k\pi}{m} \right) : \left( 1 \leq k \leq \frac{m-1}{2} \right) \land \gcd(k, m) = 1 \right\}$$

Then $A_{m,1}$ is a subset of the roots of $T_{m-1}(x) - x$.

Proof: Let $n = m - 1$. Then $n$ is even. Note that we can write $A_{m,1}$ as

$$\left\{ \cos \left( \frac{2k\pi}{n+1} \right) : \left( 1 \leq k \leq \frac{n}{2} \right) \land \gcd(k, n+1) = 1 \right\}$$

This is a subset of

$$\left\{ \cos \left( \frac{2k\pi}{n+1} \right) : \left( 1 \leq k \leq \frac{n}{2} \right) \right\}$$

By Lemma 3.1 this is a subset of the roots of $T_n(x) - x$ which is $T_{m-1}(x) - x$.  

AUGUSTE- I ADDED THE ABOVE LEMMA FOR CLARITY SINCE THE INDICES CHANGE FROM THE LAST CHAPTER TO THIS ONE.

Theorem 4.3 Let $m \in \mathbb{N}$, $m \geq 3$, $m$ odd.

AUGUSTE- I made this more parts. THE NEXT AUGUSTE comment will tell you why.

1. There exists $p_m(x) \in \mathbb{Z}[x]$ whose roots are

$$A_{m,1} = \left\{ \cos \left( \frac{2k\pi}{m} \right) : \left( 1 \leq k \leq \frac{m-1}{2} \right) \land \gcd(k, m) = 1 \right\}$$

2. All elements of $A_{m,1}$ are algebraic of degree $f_{\text{odd}}(m)$. (This follows from Part 1.)

3. There exists $q_m(x) \in \mathbb{Z}[x]$ whose roots are

$$A_{m,2} = \left\{ \cos \left( \frac{(m-2k)\pi}{m} \right) : \left( 1 \leq k \leq \frac{m-1}{2} \right) \land \gcd(k, m) = 1 \right\}$$
4. All elements of $A_{m,2}$ are algebraic of degree $f_{\text{odd}}(m)$. (This follows from Part 3.)

5. All of the numbers in
   \[ \left\{ \cos \left( \frac{k\pi}{m} \right) : (1 \leq k \leq m - 1) \land \gcd(k, m) = 1 \right\} \]
   are algebraic of degree $\leq f_{\text{odd}}(m)$. (This follows from Parts 2 and 4.)

**Proof:** To construct $p_m(x)$ we will implement the following strategy: We begin by constructing a fraction with $T_{m-1}(x) - x$ as the numerator. Note that the roots of $T_{m-1}(x) - x$ include the roots we want for $p_m(x)$. By Lemma 4.2 the roots of $p_m(x)$ are a subset of the roots of $T_{m-1}(x) - x$. In order to remove the other roots of $T_{m-1}(x) - x$ we will strategically place previously constructed polynomials, $p_{m'}(x)$, $m' < m$ in the denominator. Since all the $p_{m'}$ divide $T_{m-1}(x) - x$ and have unique roots, in the end we will be left with a polynomial with only our desired roots. Parts 2 and 3 of the above theorem will follow naturally from the construction of $p_m(x)$.

(1) We prove Part 1 by induction on $m$.

**Base Case:** $m = 3$. Then $A_{m,1} = \{ \cos(2\pi/3) \} = \{-1/2\}$ and $A_{m,2} = \{ \cos(\pi/3) \} = \{1/2\}$. Let $p_3(x) = 2x + 1$ and $q_3(x) = -2x + 1$. Both $p_3(x)$ and $q_3(x)$ are of degree $f_{\text{odd}}(3) = 1$.

**Inductive Hypothesis** The theorem is true for all odd $m'$, $3 \leq m' < m$.

**Inductive Step** Since $m$ is odd, $m - 1$ is even. By Lemma 3.1.2 the $m - 1$ roots of $T_{m-1}(x) - x = 0$ are
   \[ \left\{ \cos \left( \frac{2k\pi}{m - 2} \right) : 0 \leq k \leq \frac{m - 3}{2} \right\} \cup \left\{ \cos \left( \frac{2k\pi}{m} \right) : 1 \leq k \leq \frac{m - 1}{2} \right\}. \]

We partition this set into the following disjoint sets based on the denominators after reducing fractions.

- The $k = 0$ case yields $\cos(0) = 1$. Hence $x - 1$ divides $T_{m-1}(x) - x$.
- For all $3 \leq m' \leq m - 2$ such that $m'$ divides $m - 2$ we have:
  \[ A_{m',1} = \left\{ \cos \left( \frac{2k\pi}{m'} \right) : \left( 1 \leq k \leq \frac{m' - 1}{2} \right) \land \gcd(k, m') = 1 \right\} \]
- For all $3 \leq m' \leq m - 1$ such that $m'$ divides $m$ we have:
  \[ A_{m',1} = \left\{ \cos \left( \frac{2k\pi}{m'} \right) : \left( 1 \leq k \leq \frac{m' - 1}{2} \right) \land \gcd(k, m') = 1 \right\} \]
• The remaining roots which we want as the roots of \( p_m(x) \):

\[
A_{m,1} = \left\{ \cos\left(\frac{2k\pi}{m'}\right) : 1 \leq k \leq \frac{m-1}{2} \right\} \land \gcd(k, m) = 1
\]

Since \( m \) is odd and our \( m' \) in \( A_{m',1} \) divides either \( m - 2 \) or \( m \) we know \( m' \) is odd. Hence by our induction hypothesis there is a polynomial \( p_{m'}(x) \in \mathbb{Z}[x] \) of degree \( f_{\text{odd}}(m') \) whose roots are the elements of \( A_{m',1} \). Hence \( p_{m'} \) divides \( T_{m-1}(x) - x \).

We take the corresponding polynomials for out \( A_{m',1}s \).

BILL TO AUGUSTE: THE NOTATION \( A_{m',1}s \) Looks funny. The s at the end looks funny.

\[
\{p_{m'}(x)\}_{3 \leq m' \leq m-2} \cup \{p_{m'}(x)\}_{3 \leq m' \leq m-1} \land (m|m)
\]

Note these polynomials have disjoint sets of roots and all divide \( T_{m-1}(x) - x \). By removing these from \( T_{m-1}(x) - x \) we have that the following polynomial has exactly \( A_{m,1} \) for its roots.

\[
p_m(x) = \frac{T_{m-1}(x) - x}{(x-1)(\prod_{3 \leq m' \leq m-2, m'|m-2} p_{m'}(x))(\prod_{3 \leq m' \leq m-1, m'|m} p_{m'}(x))}
\]

BILL TO AUGUSTE: ITS NOT QUITE RIGHT TO SAY THAT THE DEGREE IS CLEARLY BLAH. ITS THE OTHER WAY AROUND- WE OBTAIN A POLY \( p_m(x) \) THAT HAS EXACTLY \( A_{m,1} \) FOR ROOTS AND THATS WHY THE DEGREE IS BLAH. I REWROTE IT THAT WAY. AND COMMENTED OUT TWO OF YOUR LINES JUST ABOVE THIS COMMENT. THIS IS WHY I CHANGED THE THEOREM TO HAVE PARTS.

3) Let \( k \) be such that \( 1 \leq k \leq \frac{m-1}{2} \) and \( \gcd(k, m) = 1 \). Note that

\[
\cos\left(\frac{(m - 2k)\pi}{m}\right) = \cos\left(-\frac{(m - 2k)\pi}{m}\right) = -\cos\left(\pi - \frac{(m - 2k)\pi}{m}\right) = -\cos\left(\frac{2k\pi}{m}\right)
\]

Hence \( A_{m,2} = -A_{m,1} \). Therefore we can take \( q_m(x) = p_m(-x) \).

**Corollary 4.4** Let \( m \in \mathbb{N}, m \geq 3, \) be odd. There exists a polynomial \( r_m(x) \in \mathbb{Z}[x] \) of degree \( m - 1 \) whose roots are

\[
\left\{ \cos\left(\frac{k\pi}{m}\right) : 1 \leq k \leq m - 1 \right\}
\]

**Proof:**

\[
r_m(x) = \prod_{m' \geq 2, m'|m} p_{m'}(x) \cdot \prod_{m' \geq 3, m'|m, 2|m} q_{m'}(x)
\]
5 A Polynomial for $\cos\left(\frac{k\pi}{m}\right)$: $m$ Even

Def 5.1 Let $m \geq 1$ be even. Define

$$f_{\text{even}}(m) = |\{k: (1 \leq k \leq m) \land \gcd(k, m) = 1\}|.\$$

$f_{\text{even}}(m)$ is the number of numbers in $\{1, \ldots, m\}$ that are relatively prime to $m$. Note that $f_{\text{even}}(m)$ is $\phi(m)$ where $\phi$ is the Euler Totient function.

BILL TO AUGUSTE: I added the above line. Its true and the reader should be told.

Theorem 5.2 Let $m \in \mathbb{N}$, $m \geq 2$, $m$ even. There exists $p_m(x) \in \mathbb{Z}[x]$ of degree $f_{\text{even}}(m)$ whose roots are

$$B_{m, 1} = \left\{\cos\left(\frac{k\pi}{m}\right): \left(1 \leq k \leq m\right) \land \gcd(k, m) = 1\right\}$$

Proof: To construct $p_m(x)$, $m$ even, we will the same strategy as the previous proof except with a different Chebyshev polynomial. We start with a fraction with $T_{2m-1}(x) - x$ as the numerator. Note that the values of $B_{m, 1}$ are roots of the polynomial. To remove the other roots we will place previously constructed polynomials $p_m'(x)$ $m' < m$ in the denominator.

1) We prove this by induction on $m$.

Base Case: $m = 2$. Then $B_{2, 1} = \{\cos(\pi/2)\} = \{0\}$. Let $p_2(x) = x$. $p_2(x)$ has degree 1.

Inductive Hypothesis The theorem is true for all even $m'$, $2 \leq m' < m$.

Inductive Step Since $m$ is even, $2m - 1$ is odd. By Lemma 3.1.2 the $2m - 1$ roots of $T_{2m-1}(x) - x = 0$ are

$$\left\{\cos\left(\frac{2k\pi}{2m - 2}\right): 0 \leq k \leq \frac{2m - 2}{2}\right\} \cup \left\{\cos\left(\frac{2k\pi}{2m}\right): 1 \leq k \leq \frac{2m - 2}{2}\right\}$$

which is

$$\left\{\cos\left(\frac{k\pi}{m - 1}\right): 0 \leq k \leq m - 1\right\} \cup \left\{\cos\left(\frac{k\pi}{m}\right): 1 \leq k \leq m - 1\right\}$$

Since $m$ is even, $m - 1$ is odd. Therefore, by Corollary 4.4, there exists $r_{m-1}(x) \in \mathbb{Z}[x]$ whose roots are the elements of the first union.

We partition the second union into the following disjoint sets based on the denominators after reducing fractions.

- The $k = 0$ case. This is just $\cos(0) = 1$. Hence $x - 1$ divides $T_{2m-1}(x) - x$.  

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• The $k = m - 1$ case. This is just $\cos(\pi) = -1$. Hence $x + 1$ divides $T_{2m-1}(x) - x$.

• For all $2 \leq m' \leq m - 1$ such that $m'$ divides $m$ we have:

$$C_{m'} = \left\{ \cos\left(\frac{k\pi}{m'}\right) : \left(1 \leq k \leq m' - 1\right) \land \gcd(k, m') = 1 \right\}.$$  

Here we have two subcases:

1. If $m'$ is odd then, by Theorem 4.3.3, there exists $p_{m'}(x) * q_{m'}(x) \in \mathbb{Z}[x]$ whose roots are $C_{m'}$. Clearly $p_{m'}(x) * q_{m'}(x)$ divides $T_{2m-1}(x) - x$.

2. If $m'$ is even then, by the induction hypothesis, there exists $p_{m'}(x) \in \mathbb{Z}[x]$ whose roots are $C_{m'}$. Clearly $p_{m'}(x)$ divides $T_{2m-1}(x) - x$.

• The remaining roots which we want as the roots of our polynomial $p_{m}(x)$:

$$B_{m,1} = \left\{ \cos\left(\frac{k\pi}{m}\right) : \left(1 \leq k \leq m\right) \land \gcd(k, m) = 1 \right\}.$$ 

Using the polynomials described above we attain the following equation with leaves us with the desired roots for pour $p_{m}(x)$

$$p_{m}(x) = \frac{T_{2m-1}(x) - x}{(x - 1)(x + 1)r_{m-1}(x) \prod_{2 \leq m' \leq m-1, m' | m} P_{m}(x) \prod_{2 \leq m' \leq m-1, m' | m, m \equiv 1 \mod 2} q_{m'}(x)}$$

Clearly the degree of this polynomial is $f_{even}(m)$.  

A  The First 11 Chebyshev Polynomials

1. $T_2(x) = 2x^2 - 1$
2. $T_3(x) = 4x^3 - 3x$
3. $T_4(x) = 8x^4 - 8x^2$
4. $T_5(x) = 16x^5 - 20x^3 + 5x$
5. $T_6(x) = 32x^6 - 48x^4 + 18x^2 - 1$
6. $T_7(x) = 64x^7 - 112x^5 + 56x^3 - 7x$
7. $T_8(x) = 128x^8 - 256x^6 + 160x^4 - 32x^2 + 1$
8. $T_9(x) = 256x^9 - 576x^7 + 432x^5 - 120x^3 + 9x$
9. \( T_{10}(x) = 512x^{10} - 1280x^8 + 1120x^6 - 400x^4 + 50x^2 - 1 \)

10. \( T_{11}(x) = 1024x^{11} - 2816x^9 + 2816x^7 - 1232x^5 - 220x^3 - 11x \)

11. \( T_{12}(x) = 2048x^{12} - 6144x^{10} + 6912x^8 - 3584x^6 + 840x^4 - 72x^2 + 1 \)

BILL TO AUGUSTE: GIVE A FEW MORE SO THAT THEY FILL UP MOST OF PAGE 7- OR STOP WHEN IT GETS INFEASIBLE.
B  Table of Polynomials

In the first column if we have a number line $\pi/4$ we mean $\cos(\pi/4)$.

<table>
<thead>
<tr>
<th>Roots</th>
<th>Poly</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\pi/2$</td>
<td>$x - 1$</td>
</tr>
<tr>
<td>$\pi/3$</td>
<td>$-2x + 1$</td>
</tr>
<tr>
<td>$2\pi/3$</td>
<td>$2x + 1$</td>
</tr>
<tr>
<td>$\pi/4, 3\pi/4$</td>
<td>$2x^2 - 1$</td>
</tr>
<tr>
<td>$\pi/5, 3\pi/5$</td>
<td>$4x^2 - 2x - 1$</td>
</tr>
<tr>
<td>$2\pi/5, 4\pi/5$</td>
<td>$4x^2 + 2x - 1$</td>
</tr>
<tr>
<td>$\pi/6, 5\pi/6$</td>
<td>$-16x^2 + 8$</td>
</tr>
<tr>
<td>$\pi/7, 3\pi/7, 5\pi/7$</td>
<td>$-8x^3 + 4x^2 + 4x - 1$</td>
</tr>
<tr>
<td>$2\pi/7, 4\pi/7, 6\pi/7$</td>
<td>$8x^3 + 4x^2 - 4x - 1$</td>
</tr>
<tr>
<td>$\pi/8, 3\pi/8, 5\pi/8, 7\pi/8$</td>
<td>$-16x^2 + 12$</td>
</tr>
<tr>
<td>$\pi/9, 5\pi/9, 7\pi/9, 9\pi/9, 4\pi/9, 8\pi/9$</td>
<td>$-8x^3 - 6x + 1$ $8x^3 - 6x + 1$</td>
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<tr>
<td>$\pi/10, 3\pi/10, 7\pi/10, 9\pi/10$</td>
<td>$16x^4 - 16x^2 + 2$</td>
</tr>
<tr>
<td>$\pi/11, 3\pi/11, 5\pi/11, 7\pi/11, 9\pi/11, 11\pi/11$</td>
<td>$-32x^4 + 16x^4 + 32x^3 - 12x^2 - 6x + 1$</td>
</tr>
<tr>
<td>$2\pi/11, 4\pi/11, 6\pi/11, 8\pi/11, 10\pi/11$</td>
<td>$32x^3 + 16x^4 - 32x^3 - 12x^2 + 6x + 1$</td>
</tr>
<tr>
<td>$\pi/12, 5\pi/12, 7\pi/12, 11\pi/12$</td>
<td>$16x^4 - 16x^2 + 1$</td>
</tr>
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<td>$\pi/13, 3\pi/13, 5\pi/13, 7\pi/13, 9\pi/13, 11\pi/13, 12\pi/13$</td>
<td>$64x^6 + 32x^3 - 80x^4 - 32x^3 + 24x^2 + 6x - 1$</td>
</tr>
<tr>
<td>$2\pi/13, 4\pi/13, 6\pi/13, 8\pi/13, 10\pi/13, 12\pi/13$</td>
<td>$64x^6 - 32x^5 - 80x^4 + 32x^3 + 24x^2 - 6x - 1$</td>
</tr>
<tr>
<td>$\pi/14, 3\pi/14, 5\pi/14, 9\pi/14, 11\pi/14, 13\pi/15, 15\pi/15$</td>
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</tr>
<tr>
<td>$\pi/15, 7\pi/15, 11\pi/15, 13\pi/15$</td>
<td>$16x^4 - 8x^3 - 16x^2 + 8x + 1$</td>
</tr>
<tr>
<td>$2\pi/15, 4\pi/15, 8\pi/15, 14\pi/15$</td>
<td>$16x^4 + 8x^3 - 16x^2 - 8x + 1$</td>
</tr>
<tr>
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</tr>
<tr>
<td>$\pi/17, 3\pi/17, 5\pi/17, 7\pi/17, 9\pi/17, 11\pi/17, 13\pi/17, 15\pi/17$</td>
<td>$256x^8 + 128x^7 - 448x^6 - 192x^5 + 240x^4 + 80x^3 - 40x^2 - 8x + 1$</td>
</tr>
<tr>
<td>$2\pi/17, 4\pi/17, 6\pi/17, 8\pi/17, 14\pi/17$</td>
<td>$256x^8 - 128x^7 - 448x^6 + 192x^5 + 240x^4 - 80x^3 - 40x^2 + 8x - 1$</td>
</tr>
</tbody>
</table>

BILL TO AUGUSTE- YOUR OTHER DOC HAS MANY MORE OF THESE. The polys begin to look nasty, even so- go until 25.

C  Using cos (2\theta) to get cos (a\pi/3)

By Lemma 3.1.2 the 2 roots of $T_2(x) - x = 0$ are

$$\{\cos(0)\} \cup \{\cos(2\pi/3)\}.$$  

Since $T_2(x) - x = 2x^2 - x - 1 = (x - 1)(2x + 1)$ we have

$$\{1, -1/2\} = \{\cos(0), \cos(2\pi/3)\}.$$
Since \( \cos(0) = 1 \), we have

1. \( \cos(2\pi/3) = -\frac{1}{2} \). Root of \( 2x + 1 \).

2. \( \cos(\pi/3) = -\cos(\pi - \pi/3) = -\cos(2\pi/3) = \frac{1}{2} \). Root of \(-2x + 1\).

**D Using \( \cos(3\theta) \) to get \( \cos(\pi/2) \)**

By Lemma 3.1.2 the 3 roots of \( T_3(x) - x = 0 \) are

\[ \{\cos(0), \cos(\pi)\} \cup \{\cos(\pi/2)\} \].

Since \( T_3(x) - x = 4x^2 - 4x = 4(x - 1)(x + 1) \)

\[ \{0, 1, -1\} = \{\cos(0), \cos(\pi/2), \cos(\pi)\} \].

We know \( \cos(0) = 1 \) and \( \cos(\pi) = -1 \), so

1. \( \cos(\pi/2) = 1 \). Root of \( x - 1 \).

**E Using \( \cos(4\theta) \) to get \( \cos(a\pi/5) \)**

By Lemma 3.1.2 the 4 roots of \( T_4(x) - x = 0 \) are

\[ \{\cos(0), \cos(2\pi/3)\} \cup \{\cos(2\pi/5), \cos(4\pi/5)\} \].

Since

\[ (1) \quad T_4(x) - x = 8x^4 - 8x^2 - x + 1 = (2x + 1)(x - 1)(4x^2 + 2x - 1) \]

(2) \( \cos(0) = 1 \), and (3) \( \cos(2\pi/3) = -1/2 \), we have that \( \cos(2\pi/5) \) and \( \cos(4\pi/5) \) are roots of \( 6x^2 + 2x - 1 \). Since \( \cos(2\pi/5) > 0 \) and \( \cos(4\pi/5) < 0 \) we have \( \cos(2\pi/5) = \frac{-1 + \sqrt{5}}{4} \) and \( \cos(4\pi/5) = \frac{-1 - \sqrt{5}}{4} \). With this we have:

1. \( \cos(\pi/5) = \frac{1 + \sqrt{5}}{4} \). Root of \( 4x^2 - 2x - 1 \).

2. \( \cos(2\pi/5) = \frac{-1 + \sqrt{5}}{4} \). Root of \( 4x^2 + 2x - 1 \).

3. \( \cos(3\pi/5) = \frac{1 - \sqrt{5}}{4} \). Root of \( 4x^2 - 2x - 1 \).

4. \( \cos(4\pi/5) = \frac{-1 - \sqrt{5}}{4} \). Root of \( 4x^2 + 2x - 1 \).
F Using cos (5θ) to get not much

By Lemma 3.1.2 the 5 roots of $T_5(x) - x = 0$ are

$$\{\cos(0), \cos(\pi/2), \cos(\pi)\} \cup \{\cos(\pi/3), \cos(2\pi/3)\}.$$ 

This will not give us any cosines we don’t already know. Darn!

Even so, we note that $T_5(x) - x = 16x^5 - 20x^3 + 4x = 4x(x - 1)(x + 1)(2x - 1)(2x + 1)$

G Using cos (6θ) to get cos (aπ/7)

By Lemma 3.1.2 the 6 roots of $T_6(x) - x = 0$ are

$$\{\cos(0), \cos(2\pi/5), \cos(4\pi/5)\} \cup \{\cos(2\pi/7), \cos(4\pi/7), \cos(6\pi/7)\}.$$ 

Since

(1) $T_6(x) - x = 32x^6 - 48x^4 + 18x^2 - x - 1 = (4x^2 + 2x - 1)(x - 1)(8x^3 + 4x^2 - 4x - 1),$

(2) $\{\cos(2\pi/5), \cos(4\pi/5)\}$ are roots of $6x^2 + 2x - 1$, and (3) $\cos(0)$ is a root of $x - 1 = 0$,

we have that

$$\{\cos(2\pi/7), \cos(4\pi/7), \cos(6\pi/7)\}$$ are the 3 roots of $8x^3 + 4x^2 - 4x - 1$.

1. $\cos(\pi/7), \cos(3\pi/7), \cos(5\pi/7)$ are roots of $-8x^3 + 4x^2 + 4x - 1$.

2. $\cos(2\pi/7), \cos(4\pi/7), \cos(6\pi/7)$ are roots of $8x^3 + 4x^2 - 4x - 1$.

H Using cos (7θ) to get cos (aπ/4)

By Lemma 3.1.2 the 7 roots of $T_7(x) - x = 0$ are

$$\{\cos(0), \cos(\pi/3), \cos(2\pi/3), \cos(\pi)\} \cup \{\cos(\pi/4), \cos(\pi/2), \cos(3\pi/4)\}.$$ 

Since

(1) $T_7(x) - x = 64x^7 - 112x^5 + 56x^3 - 8x = 8(x - 1)(x + 1)(2x - 1)(2x + 1)$$x(2x^2 - 1).$

(2) $\cos(0), \cos(\pi), \cos(\pi/3), \cos(2\pi/3), \cos(\pi/2)$ are, respectively, roots of $x - 1, x + 1, 2x - 1, 2x + 1, x,$ we have that

$$\{\cos(\pi/4), \cos(3\pi/4)\}$$ are the 2 roots of $2x^2 - 1$.

Since $\cos(\pi/4) > 0$ and $\cos(3\pi/4) < 0$ we have the following:
1. \( \cos(\pi/4) = \sqrt{2}/2 \). Root of \( 2x^2 - 1 \).

2. \( \cos(2\pi/4) = 0 \) is a root of \( x \).

3. \( \cos(3\pi/4) = -\sqrt{2}/2 \) is a root of \( 2x^2 - 1 \).

4. \( \cos(4\pi/4) = -1 \) is a root of \( x + 1 = 0 \).

**I Using \( \cos (8\theta) \) to get \( \cos (a\pi/9) \)**

By Lemma 3.1.2 the 8 roots of \( T_8(x) - x = 0 \) are

\[
\{ \cos(0), \cos(2\pi/7), \cos(4\pi/7), \cos(6\pi/7) \} \cup \{ \cos(2\pi/9), \cos(4\pi/9), \cos(2\pi/3), \cos(8\pi/9) \}.
\]

Since

(1) \( T_8(x) - x = 128x^8 - 256x^6 + 160x^4 - 32x^2 - x + 1 = (x-1)(8x^3+4x^2-4x-1)(8x^3-6x+1)(2x+1) \)

(2) \( \{ \cos(2\pi/7), \cos(4\pi/7), \cos(6\pi/7) \} \) are roots of \( 8x^3 + 4x^2 - 4x - 1 \), (3) \( \cos(0) \) is a root of \( x - 1 = 0 \), and (4) \( \cos(2\pi/3) \) is a root of \( 2x + 1 \) we have

\( \{ \cos(2\pi/9), \cos(4\pi/9), \cos(8\pi/9) \} \) are the 3 roots of \( 8x^3 - 6x + 1 \).

1. \( \cos(\pi/9), \cos(5\pi/9), \cos(7\pi/9) \) are roots of \( -8x^3 + 6x + 1 \).

2. \( \cos(2\pi/9), \cos(4\pi/9), \cos(8\pi/9) \) are roots of \( 8x^3 - 6x + 1 \).

3. \( \cos(3\pi/9) \) is a root of \( 2x - 1 \).

4. \( \cos(6\pi/9) \) is a root of \( 2x + 1 \).

**J Using \( \cos (9\theta) \) to Get Nothing New**

By Lemma 3.1.2 the 9 roots of \( T_9(x) - x = 0 \) are

\[
\{ \cos(0), \cos(\pi/4), \cos(\pi/2), \cos(3\pi/4), \cos(\pi) \} \cup \{ \cos(\pi/5), \cos(2\pi/5), \cos(3\pi/5), \cos(4\pi/5) \}.
\]

This will not give us any cosines we don’t already know. Darn! Even so, we note that

Since

\[
T_9(x) - x = 256x^9 - 576x^7 + 432x^5 - 120x^3 + 8x = (x-1)(2x^2-1)8x(x+1)(4x^2-2x-1)(4x^2+2x-1).
\]
K Using \( \cos(10\theta) \) to get \( \cos(a\pi/11) \)

By Lemma 3.1.2 the 10 roots of \( T_{10}(x) - x = 0 \) are

\[
\{\cos(0), \cos(2\pi/9), \cos(4\pi/9), \cos(2\pi/3), \cos(8\pi/9)\} \cup \\
\{\cos(2\pi/11), \cos(4\pi/11), \cos(6\pi/11), \cos(8\pi/11), \cos(10\pi/11)\}.
\]

Since

(1) \( T_{10}(x) - x = 512x^{10} - 1280x^8 + 1120x^6 - 400x^4 + 50x^2 - 1 = (x-1)(2x+1)(8x^3 - 6x + 1)(32x^5 + 16x^4 - 32x^3 - 12x^2 + 6x + 1) \)

(2) \( \cos(0) \) is a root of \( x - 1 \), (3) \( \cos(2\pi/3) \) is a root of \( 2x + 1 \), (4) \( \cos(2\pi/9), \cos(4\pi/9), \cos(8\pi/9) \) are the roots of \( 8x^3 - 6x + 1 \), we have that

\[
\{\cos(2\pi/11), \cos(4\pi/11), \cos(6\pi/11), \cos(8\pi/11), \cos(10\pi/11)\}
\]

are the 5 roots of \( 32x^5 + 16x^4 - 32x^3 - 12x^2 + 6x + 1 \).

L For \( a, b \in \mathbb{N}, \ \cos(a\pi/b) \) is Algebraic

We do not need this section anymore. If you agree than delete.

(We DO need the sections with EXAMPLES for now as a sanity check.)

The above examples show that all of the number \( \cos(\pi/2), \cos(\pi/3), \cos(\pi/4), \cos(\pi/5), \) and \( \cos(\pi/6) \) are algebraic. Are all numbers of the form \( \cos(\pi/b) \) algebraic? If so then easily so are all numbers of the form \( \cos(a\pi/b) \). The answer is Yes.

Theorem L.1

1. For all \( n \in \mathbb{N}, \ \cos(\pi/n) \) is algebraic.

2. For all \( n, m \in \mathbb{N}, \ \cos(m\pi/n) \) is algebraic. (This follows from part 1 and the cosine addition formula.)

3. For all \( n, m \in \mathbb{N}, \ \sin(m\pi/n) \) is algebraic. (This follows from part 2 and the relationship between sine and cosine, and the closure of the algebraic numbers under square roots.)

4. For all \( n, m \in \mathbb{N}, \ \tan(m\pi/n) \) is algebraic. (This follows from parts 2, 3 and the relationship between tangent, sine, cosine, and the closure of the algebraic numbers under quotients.)
Proof:
Let \( n \in \mathbb{N} \).

As noted above,
\[
T_n(\cos(n\theta)) = \cos(\theta).
\]

Let \( \theta = \frac{\pi}{n} \). Then you get
\[
T_n(\cos(\frac{\pi}{n})) = \cos(\pi) = -1.
\]

On the right hand side you get a poly in \( \cos(\frac{\pi}{n}) \) with integer coefficients. Hence \( \cos(\frac{\pi}{n}) \) is algebraic. \( \square \)

M What about Sin?

The following is an anonymous post on math stack exchange.

**Theorem M.1** \( \sin(2\theta) \) can not be written as a polynomial over \( \mathbb{R} \) in \( \sin(\theta) \).

**Proof:** Assume, by way of contradiction, that there exists a polynomial \( p(x) \in \mathbb{R}[x] \) such that \( \sin(2\theta) = p(\sin(\theta)) \). Since \( \sin(2\theta) = 2\cos(\theta)\sin(\theta) \) we have
\[
2\cos(\theta)\sin(\theta) = \sin(2\theta) = p(\sin(\theta)).
\]

Note that if \( \theta = 0 \) then the left hand side is 0, so \( p(\sin(0)) = 0 \). Hence \( p(0) = 0 \). Therefore there exists \( q(x) \in \mathbb{R}[x] \) such that \( p(x) = xq(x) \). So
\[
2\cos(\theta)\sin(\theta) = p(\sin(\theta)) = \sin(\theta)q(\sin(\theta)).
\]
\[
2\cos(\theta) = p(\sin(\theta)) = q(\sin(\theta)).
\]

(We divided by \( \sin(\theta) \) so we needed to have \( \theta \notin \{n\pi: n \in \mathbb{Z}\} \); however, by continuity the two expressions are equal for all \( \theta \).)

Square both sides and use \( \cos^2(\theta) = 1 - \sin^2(\theta) \) to get
\[
4(1 - \sin^2(\theta)) = q(\sin(\theta))^2.
\]

The two polynomials \( 4(1 - x^2) \) and \( q(x)^2 \) agree for infinitely many \( x \), namely \( \sin(\theta) \) as \( \theta \in [0, \pi] \). Hence they are equal. But \( q(x)^2 \) is a square of a polynomial, and \( 4(1 - x^2) = 4(1 - x)(1 + x) \) is not. Contradiction. \( \square \)
References
