The Algebraic Degree of $\cos(v\pi/n)$ and $\sin(v\pi/n)$
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1 Introduction

The following are well-known:

1. $\cos(\pi/1) = -1$,
2. $\cos(\pi/2) = 0$,
3. $\cos(\pi/3) = 1/2$,
4. $\cos(\pi/4) = \sqrt{2}/2$, and
5. $\cos(\pi/6) = \sqrt{3}/2$.

Note that $\cos(\pi/5)$ is missing. In Harold Boas's paper [1][1] he shows that

$$\cos\left(\frac{\pi}{5}\right) = \frac{1+\sqrt{5}}{4}$$

which is half the golden ratio. Note that all of these numbers are algebraic.

Convention 1.1

1. In this paper all variables $a, b, c, \ldots, z$ range over $\{1, 2, 3 \ldots\}$ unless otherwise noted. The variable θ ranges over R.
2. All polynomials have coefficients in Z unless otherwise noted.
3. We will be studying $\cos(v\pi/n)$ and $\sin(v\pi/n)$ where v/n is in lowest terms. Hence we will usually have as a premise that v and n are co-prime. We may also denote this by $\gcd(v, n) = 1$ when that notation is useful.

Definition 1.2 Let $d \geq 1$. Let $\alpha \in \mathsf{C}$.

1. α is *algebraic* if there exists $p \in \mathsf{Z}[x]$ such that $p(\alpha) = 0$.
2. Let α be algebraic. The *degree of* α is the least d such that there exists a $p \in \mathsf{Z}[x]$ of degree d with $p(\alpha) = 0$. We denote this by $\deg(\alpha)$. Note that we could replace Z with Q and the degree would be the same.

[1] Harold Boas's paper was the inspiration for our paper.

Are numbers of the form $\cos(v\pi/n)$ always algebraic? Yes. These statements are well-known. We will prove this for all $v \ge 1$, $n \ge 2$, $1 \le v \le n$, and $\gcd(v,n) = 1$. All other cases are either easy (e.g., $v = 0$) or can be derived from what we prove (e.g., $v \le -1$). In this paper we prove the following.

1. In Sections 2,3,4, and 5 we show $\cos(v\pi/n)$ is algebraic. The proof has the following properties:
   (a) It only uses elementary techniques.
   (b) It is self contained (with help from Appendices A and B).
   (c) We obtain upper bounds on $\deg(\cos(v\pi/n))$. The upper bounds for $\deg(\cos(v\pi/n))$ are optimal although we do not prove that.
   (d) The proof gives a way to obtain the explicit polynomials (which we do in Appendix D).
   (e) The proof may be longer than you like.
2. In Section 6 we show that $\sin(v\pi/n)$ is algebraic. In Section **??** we show that $\sec(v\pi/n)$ is algebraic. In Section **??** we show that $\tan(v\pi/n)$ is algebraic. The proofs have the following properties:
   (a) They use that results about cosine.
   (b) They are is self contained.
   (c) They obtain upper bounds on $\deg(\sin(v\pi/n))$, $\deg(\sec(v\pi/n))$, $\deg(\tan(v\pi/n))$. The upper bounds are optimal although we do not prove that.

To state our results we need the following well-known definition and theorem.

Definition 1.3 $\phi(n)$ is $|\{v\colon 1 \le v \le n-1 \text{ and } v \text{ and } n \text{ are co-prime}\}|$. This is often called *Euler's Totient Function* or the *Euler's ϕ function.*

Theorem 1.4

1. *If* $\gcd(n_1,n_2) = 1$ *then* $\phi(n_1n_2) = \phi(n_1)\phi(n_2)$.
2. *If p is a prime and* $a \ge 1$ *then* $\phi(p^a) = p^a - p^{a-1}$.
3. *If* $n \ge 3$ *then* $\phi(n)$ *is even.*
4. *If* $n \ge 3$ *then*

$$\frac{1}{2}\phi(n) = \left|\left\{k\colon 1 \le k \le \left\lfloor \frac{n-1}{2} \right\rfloor \quad \textit{and} \quad \gcd(k,n) = 1\right\}\right|.$$

(To prove this use that $\gcd(x,n) = \gcd(n-x,n)$ *and that* $f(x) = n - x$ *is a bijection from* $\{1,\ldots,\lfloor\frac{n-1}{2}\rfloor\}$ *to* $\{\lfloor\frac{n-1}{2}\rfloor + 1,\ldots,n-1\}$.*)*

This paper's contents are as follows.

1. In Section 2, we define the Chebyshev polynomials of the first kind, T_n, and state the well-known theorem about them: $\cos(nx) = T_n(\cos(x))$. We also state a theorem about dividing Chebyshev polynomials by other polynomials. Both of these theorems are proven in the appendix.

2. In Section 3, we show that, for all v, n with $1 \le v \le n$, $\deg(\cos(v\pi/n)) \le 2n + 1$. We also prove lemmas that are used in the next two sections to obtain better upper bounds on $\deg(\cos(v\pi/n))$.

3. In Section 4, we show the following: For all v, n, n odd, $1 \le v \le n-1$, and $\gcd(v, n) = 1$, $\deg(\cos(v\pi/n)) \le \phi(n)/2$. Our proof gives a construction of the needed polynomials.

4. In Section 5, we show the following: For all v, n, n even, $1 \le v \le n-1$, and $\gcd(v, n) = 1$, $\deg(\cos(v\pi/n)) \le \phi(n)$. Our proof gives a construction of the needed polynomials.

5. In Section 6, we show the following:

   (a) If $n \equiv 0 \pmod 4$ then $\deg(\sin(v\pi/n)) \le \phi(n)$.

   (b) If $n \equiv 2 \pmod 4$ then $\deg(\sin(v\pi/n)) \le \phi(n/2)/2$.

   (c) If n is odd then $\deg(\sin(v\pi/n)) \le \phi(2n)$.

6. In Appendix A we prove that, for all n, $\cos(nx) = T_n(\cos(x))$. This is well-known and proven here for the sake of completeness.

7. In Appendix B we prove that, if $p \in \mathsf{Z}[x]$ and all of the roots of p are also roots of T_n, then $T_n(x)/p(x) \in \mathsf{Z}[x]$. This is surely known; however, we could not find a proof it it.

8. In Appendix C, we list the first 39 Chebyshev polynomials of the first kind. We need these for the next Appendix.

9. In Appendix D, we give, for $1 \le v < m \le 21$, $\gcd(k, n) = 1$, a polynomial $p \in \mathsf{Z}[x]$ such that $p(\cos(v\pi/n)) = 0$. If n is odd then the polynomial has degree $\phi(n)/2$. If n is even then the polynomial has degree $\phi(n)$.

2 Chebyshev Polynomials of the First Kind

Definition 2.1 The *Chebyshev polynomials of the first kind,* T_n, $n \ge 1$, are defined by

$$T_n(x) = \sum_{k=0}^{\lfloor n/2 \rfloor} \binom{n}{2k} (x^2 - 1)^k x^{n-2k}.$$

We will need the next three theorems.

The following theorem is well-known; however, we provide a proof in Appendix A for completeness.

Theorem 2.2 *For all n, $T_n(\cos(\theta)) = \cos(n\theta)$.*

The following is surely known; however, we could not find it anywhere. We provide a prove in Appendix B (Theorem B.4).

Theorem 2.3 *Let $n \geq 1$. Let $p \in \mathsf{Z}[x]$. If the set of roots of p is a subset of the set of roots of T_n then $T_n(x)/p(x) \in \mathsf{Z}[x]$.*

For the following theorem (1) the first two parts are obvious, and (2) the third part we will prove in Lemma 3.1.

Theorem 2.4

1. *The polynomial $T_1(x) - x$ is identically 0, and hence has an infinite number of roots.*
2. *For $n \geq 2$, T_n is a polynomial of degree n.*
3. *T_n has n distinct roots.*

Since we called these *Chebyshev polynomials of the first kind* the reader may wonder if there are *Chebyshev polynomials of the second kind* and, if so, what properties they have. The *Chebyshev polynomials of the second kind*, U_n, have the following properties:

- $U_n \in \mathsf{Z}[x]$,
- U_n has degree n,
- $U_n(\cos\theta)\sin\theta = \sin((n+1)\theta)$.

We will not be using these polynomials.

3 $\deg(\cos(v\pi/n)) \leq 2n+1$

Lemma 3.1

1. *Let $n \geq 1$. For all*
$$\theta \in \left\{\frac{2k\pi}{n-1} : k \in \mathsf{Z}\right\} \cup \left\{\frac{2k\pi}{n+1} : k \in \mathsf{Z}\right\},$$
$\cos(\theta) = \cos(n\theta)$. *(If $n = 1$, then just use the second unionand.)*

2. If n is odd and $n \geq 3$, then the n roots of $T_n(x) - x$ are

$$\left\{\cos\left(\frac{2k\pi}{n-1}\right) : 0 \leq k \leq \frac{n-1}{2}\right\} \cup \left\{\cos\left(\frac{2k\pi}{n+1}\right) : 1 \leq k \leq \frac{n-1}{2}\right\}.$$

3. If n is even and $n \geq 2$, then the n roots of $T_n(x) - x$ are

$$\left\{\cos\left(\frac{2k\pi}{n-1}\right) : 0 \leq k \leq \frac{n-2}{2}\right\} \cup \left\{\cos\left(\frac{2k\pi}{n+1}\right) : 1 \leq k \leq \frac{n}{2}\right\}.$$

Proof:
1) For the first unionand notice that

$$\cos\left(\frac{2k\pi}{n-1}\right) = \cos\left(\frac{2k\pi}{n-1} + 2k\pi\right) = \cos\left(\frac{n2k\pi}{n-1}\right).$$

For the second unionand notice that

$$\cos\left(\frac{2k\pi}{n+1}\right) = \cos\left(-\frac{2k\pi}{n+1}\right) = \cos\left(2\pi k - \frac{2k\pi}{n+1}\right) = \cos\left(\frac{n2k\pi}{n+1}\right).$$

2) Let

$$X = \left\{\cos\left(\frac{2k\pi}{n-1}\right) : 0 \leq k \leq \frac{n-1}{2}\right\} \cup \left\{\cos\left(\frac{2k\pi}{n+1}\right) : 1 \leq k \leq \frac{n-1}{2}\right\}.$$

By Theorem 2.2 and Part 1 we have that all of the elements in X are roots of $T_n(x) - x$. By algebra one can see that all of the angles mentioned in the definition of X are distinct and in $[0, \pi]$. Since cosine is injective on $[0, \pi]$, X contains n different numbers. Since $n \geq 2$, by Theorem 2.4, $T_n(x) - x$ has n roots. The elements of X are its n roots.

3) Similar to the proof of Part 2. ∎

Lemmas 3.1.2 and 3.1.3 imply many cosines are algebraic.

Example 3.2
We look at the first unionand in the $n = 3, 5, 7$ cases of Lemma 3.1.2.

1. $n = 3$.
$$\left\{\cos\left(\frac{2k\pi}{2}\right) : 0 \leq k \leq 1\right\} = \{\cos(0), \cos(\pi)\}$$

2. $n = 5$.
$$\left\{\cos\left(\frac{2k\pi}{4}\right) : 0 \leq k \leq 2\right\} = \{\cos(0), \cos(\pi/2), \cos(\pi)\}$$

3. $n = 7$.

$$\left\{\cos\left(\frac{2k\pi}{6}\right) : 0 \le k \le 3\right\} = \{\cos(0), \cos(\pi/3), \cos(2\pi/3), \cos(\pi)\}$$

Theorem 3.3

1. *There exists a polynomial in* $\mathsf{Z}[x]$ *of degree* $2n+1$ *that has roots*

$$\left\{\cos\left(\frac{v\pi}{n}\right) : 0 \le v \le n\right\}.$$

2. *Let* v, n *be such that* $n \ge 1$ *and* $0 \le v \le n$. *Then* $\deg(\cos(v\pi/n)) \le 2n+1$. *(This follows from Part 1.)*

Proof: By Lemma 3.1.2, applied to $2n+1$ (since $n \ge 1$, $2n+1 \ge 3$) and replacing v with k, the elements of

$$\left\{\cos\left(\frac{2v\pi}{2n}\right) : 0 \le v \le \frac{2n}{2}\right\} = \left\{\cos\left(\frac{v\pi}{n}\right) : 0 \le v \le n\right\}.$$

are roots of $T_{2n+1}(x) - x$. Since the degree of T_{2n+1} is $2n+1$, $\deg(\cos(v\pi/n)) \le 2n+1$. ∎

4 If n is Odd Then $\deg(\cos(v\pi/n)) \le \frac{1}{2}\phi(n)$

In this section:

1. We will prove that, for all $1 \le v \le 22$, $\gcd(v, 27) = 1$, $\deg(\cos(v\pi/27)) \le \frac{1}{2}\phi(27) = 9$. We use 27 since it is the least odd number x such that both x and $x-2$ are not primes. This is important since if x or $x-2$ are primes then part of the proof is easy and will not demonstrate aspects of the general theorem. The proof will use an inductive assumption.

2. We will prove that, for all n, for all $1 \le v \le n-1$, n odd, $\gcd(v, n) = 1$, $\deg(\cos(v\pi/n)) \le \frac{1}{2}\phi(n)$. The proof is by induction.

4.1 An Example: $\deg(\cos(v\pi/27))$

The general proof constructs two polynomials $c_{n,o,1}$ and $c_{n,o,2}$, inductively on n. (c stands for cosine, o stands for odd.) The union of the roots of $c_{n,o,1}$ and $c_{n,o,2}$ are

$$\left\{\cos\left(\frac{v\pi}{n}\right) : 1 \le v \le n-1 \text{ and } \gcd(v, n) = 1\right\}.$$

As an example we will show how the induction step works to give $\mathrm{co}_{27,1}$, $\mathrm{co}_{27,2}$ For purposes of the example, we will assume we have $\mathrm{co}_{\mathrm{n}',1}$ and $\mathrm{co}_{\mathrm{n}',2}$ for all $1 \le n' \le 26$.

We show the following:

Every element of

$$\mathrm{CO}_{27} = \left\{ \cos\left(\frac{\mathrm{v}\pi}{27}\right) : 1 \le \mathrm{v} \le 26 \quad \textit{and} \quad \gcd(\mathrm{v}, 27) = 1 \right\},$$

(CO stands for cosine.)
has degree $\le \phi(27)/2 = 9$.

We construct two polynomials $\mathrm{co}_{27,1}, \mathrm{co}_{27,2} \in \mathsf{Z}[\mathrm{x}]$ of degree $\phi(27)/2 = 9$ such that the following hold.

1. The roots of $\mathrm{co}_{27,1}$ are all $\cos(v\pi/27)$ where $1 \le v \le 26$, $\gcd(v, 27) = 1$, and v is even. Formally:

$$\mathrm{CO}_{27,1} = \left\{ \cos\left(\frac{2\mathrm{k}\pi}{27}\right) : 1 \le \mathrm{k} \le 13 \quad \text{and} \quad \gcd(\mathrm{k}, 27) = 1 \right\}$$

$$= \left\{ \cos\left(\frac{2\pi}{27}\right), \cos\left(\frac{4\pi}{27}\right), \cos\left(\frac{8\pi}{27}\right), \cos\left(\frac{10\pi}{27}\right) \right\}$$

$$\cup \quad \left\{ \cos\left(\frac{14\pi}{27}\right), \cos\left(\frac{16\pi}{27}\right), \cos\left(\frac{20\pi}{27}\right), \cos\left(\frac{22\pi}{27}\right), \cos\left(\frac{26\pi}{27}\right) \right\}.$$

2. The roots of $\mathrm{co}_{27,2}$ are all $\cos(v\pi/27)$ where $1 \le v \le 26$, $\gcd(v, 27) = 1$, and v is odd. Formally:

$$\mathrm{CO}_{27,2} = \left\{ \cos\left(\frac{(27 - 2\mathrm{k})\pi}{27}\right) : 1 \le \mathrm{k} \le 13 \quad \text{and} \quad \gcd(\mathrm{k}, 27) = 1 \right\}$$

$$= \left\{ \cos\left(\frac{\pi}{27}\right), \cos\left(\frac{5\pi}{27}\right), \cos\left(\frac{7\pi}{27}\right), \cos\left(\frac{11\pi}{27}\right), \cos\left(\frac{13\pi}{27}\right) \right\}$$

$$\cup \quad \left\{ \cos\left(\frac{17\pi}{27}\right), \cos\left(\frac{19\pi}{27}\right), \cos\left(\frac{23\pi}{27}\right), \cos\left(\frac{25\pi}{27}\right) \right\}.$$

1) We construct $\mathrm{co}_{27,1}$ with an inductive assumption.

Assume that, for $3 \le n' \le 25$, n' odd, there exists $\mathrm{co}_{\mathrm{n}',1} \in \mathsf{Z}[\mathrm{x}]$ of degree $\phi(n')/2$ whose roots are

$$CO_{n',1} = \left\{ \cos\left(\frac{2k\pi}{n'}\right) : 1 \le k \le \frac{n'-1}{2} \quad \text{and} \quad \gcd(k, n') = 1 \right\}.$$

To construct $\mathrm{co}_{27,1}$ we first take $T_{26}(x) - x$. By Lemma 3.1.3 the roots of $T_{26}(x) - x$ are

$$\left\{\cos\left(\frac{2k\pi}{25}\right) : 0 \le k \le \frac{24}{2}\right\} \cup \left\{\cos\left(\frac{2k\pi}{27}\right) : 1 \le k \le \frac{26}{2}\right\}$$
$$= \left\{\cos\left(\frac{2k\pi}{25}\right) : 0 \le k \le 12\right\} \cup \left\{\cos\left(\frac{2k\pi}{27}\right) : 1 \le k \le 13\right\}.$$

Note that $CO_{27,1}$ is a subset of the roots of $T_{26}(x) - x$. To remove the other roots we will divide $T_{26}(x) - x$ by some polynomials. We will partition the roots $\cos(v\pi/n')$ (with v/n' in lowest terms) that we want to get rid of into groups. Each group will have the same n'. For example, one of the groups is $\{\cos(2\pi/5), \cos(4\pi/5)\}$. For each group there will be a polynomial that has exactly the elements of that group for roots

1. $\{\cos(0\pi/25)\} = \{\cos(0)\} = \{1\}$.

   The polynomial $x - 1$ of degree 1 suffices.

2. $$\{\cos(2\pi/25), \cos(4\pi/25), \cos(6\pi/25), \cos(8\pi/25), \cos(12\pi/25)\} \quad \cup$$
   $$\{\cos(14\pi/25), \cos(16\pi/25), \cos(18\pi/25), \cos(22\pi/25), \cos(24\pi/25)\}$$

   which is
   $$\{\cos(2k\pi/25) \colon 1 \le k \le 12 \ \text{ and } \ \gcd(k, 25) = 1\}.$$

   By assumption with $n' = 25$, there is a polynomial $\mathrm{co}_{25,1} \in \mathsf{Z}[\mathrm{x}]$ of degree $\phi(25)/2 = 10$, whose roots are this set.

3. $\{\cos(10\pi/25), \cos(20\pi/25)\} = \{\cos(2\pi/5), \cos(4\pi/5)\}$ which is

   $$\{\cos(2k\pi/5) \colon 1 \le k \le 2 \ \text{ and } \ \gcd(k, 5) = 1\}.$$

   By assumption with $n' = 5$, there is a polynomial $\mathrm{co}_{5,1} \in \mathsf{Z}[\mathrm{x}]$ of degree $\phi(5)/2 = 2$, whose roots are this set.

4. $\{\cos(6\pi/27), \cos(12\pi/27), \cos(24\pi/27)\} = \{\cos(2\pi/9), \cos(4\pi/9), \cos(8\pi/9)\}$ which is

   $$\{\cos(2k\pi/9) \colon 1 \le k \le 4 \ \text{ and } \ \gcd(k, 9) = 1\}.$$

   By assumption with $n' = 9$, there is a polynomial $\mathrm{co}_{9,1} \in \mathsf{Z}[\mathrm{x}]$ of degree $\phi(9)/2 = 3$, whose roots are this set.

5. $\{\cos(18\pi/27)\} = \{\cos(2\pi/3)\}$ which is

   $$\{\cos(2k\pi/3) \colon 1 \le k \le 1 \ \text{ and } \ \gcd(k, 3) = 1\}.$$

By assumption with $n' = 3$, there is a polynomial $\mathrm{co}_{3,1} \in \mathsf{Z}[\mathrm{x}]$ of degree $\phi(3) = 1$, whose roots are this set. (Since $\cos(2\pi/3) = -1/2$ we know that $\mathrm{co}_2(\mathrm{x}) = 2\mathrm{x} + 1$; however, by using the assumption this derivation of $\mathrm{co}_{27,1}$ is similar to the proof of the general theorem in the next subsection.)

The set of roots of $T_{26}(x) - x$ that are not in $CO_{27,1}$ is the set of roots of one of the five polynomials above. Hence we take

$$\mathrm{co}_{27,1}(\mathrm{x}) = \frac{\mathrm{T}_{26}(\mathrm{x}) - \mathrm{x}}{(\mathrm{x}-1)\mathrm{co}_{25,1}(\mathrm{x})\mathrm{co}_{5,1}(\mathrm{x})\mathrm{co}_{9,1}\mathrm{co}_{3,1}(\mathrm{x})}.$$

$\mathrm{co}_{27,1} \in \mathsf{Z}[\mathrm{x}]$ by Theorem 2.3. The set of roots of $\mathrm{co}_{27,1}(\mathrm{x})$ is $CO_{27,1}$.

As a sanity check we calculate the degree of $\mathrm{co}_{27,1}$ based on the degrees of the numerator and denominator in the definition of $\mathrm{co}_{27,1}$. The degree of $\mathrm{co}_{27,1}$ is

$$\deg(T_{26}(x) - x) - \deg(x-1) - \deg(co_{25,1}) - \deg(co_{5,1}) - \deg(co_{9,1}) - \deg(co_{3,1})$$

$$= 26 - 1 - 10 - 2 - 3 - 1 = 9.$$

This passes the sanity check since $\mathrm{co}_{27,1}(\mathrm{x})$ is supposed to have 9 roots.

2) We construct the polynomial $\mathrm{co}_{27,2} \in \mathsf{Z}[\mathrm{x}]$ by using $\mathrm{co}_{27,1}$. Note that

$$\cos\left(\frac{(27-2k)\pi}{27}\right) = \cos\left(\frac{-(27-2k)\pi}{27}\right) = -\cos\left(\pi - \frac{-(27-2k)\pi}{27}\right) = -\cos\left(\frac{2k\pi}{27}\right).$$

Hence every element in $CO_{27,2}$ is the negation of an element in $CO_{27,1}$ and vice versa. Hence we can take $\mathrm{co}_{27,2}(\mathrm{x}) = \mathrm{co}_{27,1}(-\mathrm{x})$. Clearly $f_{27,2}$ is of degree $\phi(27)/2 = 9$.

We have constructed $\mathrm{co}_{27,1}$ and $\mathrm{co}_{27,2}$ as promised.

4.2 General Theorem: If n is Odd Then $\deg\left(\cos\left(v\pi/n\right)\right) \le \frac{1}{2}\phi(n)$

Lemma 4.1 *Let $n \ge 3$, n odd. Let*

$$CO_{n,1} = \left\{\cos\left(\frac{2k\pi}{n}\right) : 1 \le k \le \frac{n-1}{2} \quad \textit{and} \quad \gcd(k,n) = 1\right\}.$$

Then $CO_{n,1}$ is a subset of the roots of $T_{n-1}(x) - x$.

Proof: Since n is odd, $n-1$ is even. By Lemma 3.1.2, applied to $n-1$, the $n-1$ roots of $T_{n-1}(x) - x$ are

$$\left\{\cos\left(\frac{2k\pi}{n-2}\right) : 0 \le k \le \frac{n-3}{2}\right\} \cup \left\{\cos\left(\frac{2k\pi}{n}\right) : 1 \le k \le \frac{n-1}{2}\right\}.$$

Clearly $CO_{n,1}$ is a subset of this set. ∎

Theorem 4.2 *Let $n \geq 3$, n odd.*

1. *There is a polynomial* $\mathrm{co_{n,1}} \in \mathsf{Z}[\mathrm{x}]$ *of degree* $\phi(n)/2$ *whose roots are*
$$CO_{n,1} = \left\{ \cos\left(\frac{2k\pi}{n}\right) : 1 \leq k \leq \frac{n-1}{2} \quad \textit{and} \quad \gcd(k,n) = 1 \right\}.$$

2. *There is a polynomial* $\mathrm{co_{n,2}} \in \mathsf{Z}[\mathrm{x}]$ *of degree* $\phi(n)/2$ *whose roots are*
$$CO_{n,2} = \left\{ \cos\left(\frac{(n-2k)\pi}{n}\right) : 1 \leq k \leq \frac{n-1}{2} \quad \textit{and} \quad \gcd(k,n) = 1 \right\}.$$

3. *Every element of*
$$CO_{n} = \left\{ \cos\left(\frac{v\pi}{n}\right) : 1 \leq v \leq n-1 \quad \textit{and} \quad \gcd(v,n) = 1 \right\}$$
has degree $\leq \phi(n)/2$.

Proof:
1) We construct $\mathrm{co_{n,1}}$ by induction on n.
Base Case: $n = 3$. Then $CO_{3,1} = \{\cos(2\pi/3)\} = \{-1/2\}$. Let $\mathrm{co_{3,1}(x) = 2x + 1}$. Note that $\mathrm{co_{3,1}}$ is of degree $\phi(3)/2 = 1$.
Induction Hypothesis Assume $n \geq 5$ is odd. Assume that, for all $1 \leq n' < n$, n odd, there exists a polynomial $\mathrm{co_{n',1}} \in \mathsf{Z}[\mathrm{x}]$ of degree $\leq \phi(n')/2$ whose roots are
$$CO_{n',1} = \left\{ \cos\left(\frac{2k\pi}{n'}\right) : 1 \leq k \leq \frac{n'-1}{2} \quad \text{and} \quad \gcd(k,n') = 1 \right\}.$$

Induction Step To construct $\mathrm{co_{n,1}}$ we first take $T_{n-1}(x) - x$. By Lemma 3.1.3 the $n-1$ roots of $T_{n-1}(x) - x$ are
$$\left\{ \cos\left(\frac{2k\pi}{n-2}\right) : 0 \leq k \leq \frac{n-3}{2} \right\} \cup \left\{ \cos\left(\frac{2k\pi}{n}\right) : 1 \leq k \leq \frac{n-1}{2} \right\}.$$

By Lemma 4.1 $CO_{n,1}$ is a subset of the roots of $T_{n-1}(x) - x$. To remove the other roots we will divide $T_{n-1}(x) - x$ by some polynomials. We list sets of roots and the polynomial that has exactly that set of roots. We also include degrees for a sanity check. For that purpose we point out that the degree of $T_{n-1}(x) - x$ is $n-1$.

1. $\{\cos(0\pi/n)\} = \{\cos(0)\} = \{1\}$.

   The polynomial $x - 1$ of degree 1 suffices.

2. For all $3 \leq n' \leq n-2$ such that n' divides $n-2$ let

$$CO_{n',1} = \left\{ \cos\left(\frac{2k\pi}{n'}\right) : 1 \leq k \leq \frac{n'-1}{2} \text{ and } \gcd(k,n') = 1 \right\}.$$

   By the inductive hypothesis there is a polynomial $\mathrm{co}_{n',1} \in \mathsf{Z}[\mathrm{x}]$ of degree $\phi(n')/2$ whose roots are this set.

3. Let $Q(n)$ be the product of all the $\mathrm{co}_{n',1}$ where $3 \leq n' \leq n-2$ and n' divides $n-2$.

4. For all $3 \leq n' \leq n-1$ such that n' divides n we have:

$$CO_{n',1} = \left\{ \cos\left(\frac{2k\pi}{n'}\right) : 1 \leq k \leq \frac{n'-1}{2} \text{ and } \gcd(k,n') = 1 \right\}.$$

   By the inductive hypothesis there is a polynomial $\mathrm{co}_{n',1} \in \mathsf{Z}[\mathrm{x}]$ of degree $\phi(n')/2$ whose roots are this set.

5. Let $R(n)$ be the product of all the $\mathrm{co}_{n',1}$ where $3 \leq n' \leq n-2$ and n' divides n.

The set of roots of $T_{26}(x)-x$ that are not in $CO_{n,1}$ is the set of roots of $(x-1)Q_{n,1}(x)R_{n,1}(x)$. Hence we take

$$co_{n,1}(x) = \frac{T_{n-1}(x) - x}{(x-1)Q_{n,1}(x)R_{n,1}(x)}.$$

$\mathrm{co}_{n,1} \in \mathsf{Z}[\mathrm{x}]$ by Theorem 2.3. The set of roots of $\mathrm{co}_{n,1}(\mathrm{x})$ is $CO_{n,1}$.
Since the roots of $\mathrm{co}_{n,1}$ are $CO_{n,1}$, the degree of $\mathrm{co}_{n,1}$ is $|CO_{n,1}| = \phi(n)/2$.

2) We construct the polynomial $\mathrm{co}_{n,2} \in \mathsf{Z}[\mathrm{x}]$ by using $\mathrm{co}_{n,1}$. Note that

$$\cos\left(\frac{(n-2k)\pi}{n}\right) = \cos\left(\frac{-(n-2k)\pi}{n}\right) = -\cos\left(\pi - \frac{-(n-2k)\pi}{n}\right) = -\cos\left(\frac{2k\pi}{n}\right).$$

Hence every element in $CO_{n,2}$ is the negation of an element in $CO_{n,1}$ and vice versa. Hence we can take $\mathrm{co}_{n,2}(\mathrm{x}) = \mathrm{co}_{n,1}(-\mathrm{x})$. Clearly $f_{n,2}$ is of degree $\phi(n)/2$.
We have constructed $\mathrm{co}_{n,1}$ and $\mathrm{co}_{n,2}$ as promised.

3) It is easy to show that $CO_n = CO_{n,1} \cup CO_{n,2}$. We leave this proof to the reader.
Since $CO_n = CO_{n,1} \cup CO_{n,2}$ we have, for every element $\alpha \in CO_n$, a polynomial of degree $\phi(n)/2$ with root α. ∎

Corollary 4.3 *Let $n \geq 3$ be odd.*

1. *There exists a polynomial* $\mathrm{co_n} \in \mathsf{Z}[\mathrm{x}]$ *of degree* $n-1$ *whose roots are*

$$\left\{\cos\left(\frac{v\pi}{n}\right) : 1 \le v \le n-1\right\}.$$

2. *There exists a polynomial* $s_n \in \mathsf{Z}[x]$ *of degree* $\phi(n)$ *such that*
   - *The roots of* s_n *are*

$$\left\{\cos\left(\frac{v\pi}{n}\right) : 1 \le v \le n-1 \quad and \quad \gcd(v,n) = 1\right\}.$$

   - *Every monomial of* s_n *is of even degree. Hence there exists* $q_n \in \mathsf{Z}[x]$ *of degree* $\phi(n)/2$ *such that* $s_n(x) = q_n(x^2)$.

Proof:

Let $\mathrm{co_{n,1}}$ and $\mathrm{co_{n,2}}$ be as in Theorem 4.2.

1)

$$co_n(x) = \prod_{n'\ge 2, n'|n} co_{n',1}(x). \prod_{n'\ge 3, n'|n, 2\nmid n} co_{n',1}(x)$$

2) $s_n(x) = co_{n,1}(x)co_{n,2}(x)$. The proof of Theorem 4.2 shows that the first two properties hold . We prove the third property.

The roots of s_n can be partitioned into $\phi(n)/2$ sets of size 2 as follows.

For $1 \le v \le n$ such that $\gcd(v,n) = 1$ we have part

$$P_n = \left\{\cos\left(\frac{v\pi}{n}\right), \cos\left(\frac{(n-v)\pi}{n}\right)\right\} = \left\{\cos\left(\frac{v\pi}{n}\right), -\cos\left(\frac{(v\pi}{n}\right)\right\}$$

Let the roots be $\alpha_1, -\alpha_1, \alpha_2, -\alpha_2, \ldots, \alpha_e, -\alpha_e$ where $e = \phi(n)/2$. Then there exists $a \in \mathsf{Q}$ such that

$$s_n(x) = a(x+\alpha_1)(x-\alpha_1)\cdots(x+\alpha_e)(x-\alpha_e).$$

Clearly the monomials of s_n all have even degree. ∎

5 If n is Even Then $\deg(\cos(v\pi/n)) \le \phi(n)$

In this section:

1. We will prove that, for all $1 \le v \le 17$, $\gcd(v,18) = 1$, $\deg(\cos(v\pi/18)) \le \phi(18) = 6$. We use 18 since it is the least even number a that has an non-prime odd factor. This is important since if a only has prime odd factors then part of the proof is easy and will not demonstrate aspects of the general theorem. The proof will use an inductive assumption.

2. We will prove that, for all n, for all $1 \le v \le n-1$, n even, $\gcd(v,n) = 1$, $\deg(\cos(v\pi/n)) \le \phi(n)$. The proof is by induction.

5.1 An Example: $\deg\left(\cos\left(v\pi/18\right)\right)$

We will do an example of the general proof which is in the next subsection. We will show (given an inductive assumption) the following:

Every element of

$$\mathrm{CO}_{18} = \left\{ \cos\left(\frac{\mathrm{v}\pi}{18}\right) : 1 \le \mathrm{v} \le 17 \quad \textit{and} \quad \gcd(\mathrm{v}, 18) = 1 \right\}$$

$$= \left\{ \cos\left(\frac{\pi}{18}\right), \cos\left(\frac{5\pi}{18}\right), \cos\left(\frac{7\pi}{18}\right), \cos\left(\frac{11\pi}{18}\right), \cos\left(\frac{13\pi}{18}\right), \cos\left(\frac{17\pi}{18}\right) \right\},$$

has degree $\le \phi(18) = 6$.

We construct $p_{18} \in \mathsf{Z}[x]$ of degree $\phi(18) = 6$ whose roots are CO_{18}.

We construct p_{18} with an inductive assumption.

Assume that, for $2 \le n' \le 16$, n' even, there exists $p_{n'} \in \mathsf{Z}[x]$ of degree $\phi(n')$ whose roots are

$$\mathrm{CO}_{\mathrm{n}'} = \left\{ \cos\left(\frac{\mathrm{v}\pi}{\mathrm{n}'}\right) : 1 \le \mathrm{v} \le \mathrm{n}' - 1 \quad \text{and} \quad \gcd(\mathrm{v}, \mathrm{n}') = 1 \right\}.$$

To construct p_{18} we first take $T_{35}(x) - x$. By Lemma 3.1.2 the roots of $T_{35}(x) - x$ are

$$\left\{ \cos\left(\frac{2v\pi}{34}\right) : 0 \le v \le \frac{34}{2} \right\} \cup \left\{ \cos\left(\frac{2v\pi}{36}\right) : 1 \le v \le \frac{34}{2} \right\}$$

$$= \left\{ \cos\left(\frac{v\pi}{17}\right) : 0 \le v \le 17 \right\} \cup \left\{ \cos\left(\frac{v\pi}{18}\right) : 1 \le v \le 17 \right\}.$$

Note that CO_{18} is a subset of the roots of $T_{35}(x) - x$. To remove the other roots we will divide $T_{35}(x) - x$ by some polynomials. We will partition the roots $\cos(v\pi/n')$ (with v/n' in lowest terms) that we want to get rid of into groups. Each group will have the same n'. For example, one of the groups is $\{\cos(\pi/6), \cos(5\pi/6)\}$. For each group there will be a polynomial that has exactly the elements of that group for roots

1. $\{\cos(0\pi/17)\} = \{\cos(0)\} = \{1\}$.

   The polynomial $x - 1$ of degree 1 suffices.

2. $\{\cos(17\pi/17)\} = \{\cos(\pi)\} = \{-1\}$.

   The polynomial $x + 1$ of degree 1 suffices.

3. $\{\cos(k\pi/17) : 1 \le k \le 16\}$.

   By Corollary 4.3.1 there is a polynomial r_{17} of degree $17 - 1 = 16$ whose roots are this set.

(In the general proof we will have a similar case where we look at $\{\cos(k\pi/(n-1)) \colon 1 \le k \le n-2\}$. There will be a polynomial that has exactly this set for its root. This will also hold when $n-1$ is not a prime. That is, the fact that 17 is a prime is not the reason why this case worked out to use just one polynomial.)

4. $\{\cos(\pi/9), \cos(2\pi/9), \cos(4\pi/9), \cos(5\pi/9), \cos(7\pi/9), \cos(8\pi/9)\}$

   which is $\{\cos(k\pi/9) \colon 1 \le k \le 8 \ \text{ and } \ \gcd(k,9)=1\}$.

   By Corollary 4.3.2 with $n'=9$, there is a polynomial $s_9 \in \mathsf{Z}[x]$ of degree $\phi(9)=6$, whose roots are this set.

5. $\{\cos(\pi/6), \cos(5\pi/6)\}$

   which is $\{\cos(k\pi/6) \colon 1 \le k \le 5 \ \text{ and } \ \gcd(k,6)=1\}$.

   By assumption with $n'=6$, there is a polynomial $p_6 \in \mathsf{Z}[x]$ of degree $\phi(6)=2$, whose roots are this set.

6. $\{\cos(\pi/3), \cos(2\pi/3)\}$.

   By Corollary 4.3.2 with $n'=3$ there is a polynomial $s_3 \in \mathsf{Z}[x]$ of degree $\phi(3)=2$ whose roots are this set.

7. $\{\cos(\pi/2)\}$

   which is $\{\cos(k\pi/2) \colon 1 \le k \le 1 \ \text{ and } \ \gcd(k,2)=1\}$.

   By assumption with $n'=2$, there is a polynomial $p_2 \in \mathsf{Z}[x]$ of degree $\phi(2)=1$,)hose roots are this set. (Since $\cos(\pi/2)=0$ we know that $p_2(x)=x$; however, by using the assumption this derivation of p_{18} is similar to the proof of the general theorem in the next subsection.)

All of the roots of $T_{35}(x)-x$ that are not in CO_{18} are roots of one of the seven polynomials above. Hence we take

$$p_{18}(x) = \frac{T_{35}(x)-x}{(x-1)(x+1)r_{17}(n)s_9(x)p_6(x)s_3(x)p_2(x)}.$$

$p_{18} \in \mathsf{Z}[x]$ by Theorem 2.3. The set of roots of $p_{18}(x)$ is CO_{18}.

As a sanity check we calculate the degree of p_{18} based on the degrees of the numerator and denominator in the definition of p_{18}. The degree of $p_{18}(x)$ is

$$\deg(T_{35}(x)-x) - \deg(x-1) - \deg(x+1) - \deg(r_{17}) - \deg(s_9) - \deg(p_6) - \deg(s_3) - \deg(p_2)$$

$$= 35 - 1 - 1 - 16 - 6 - 2 - 2 - 1 = 6.$$

This passes the sanity check since $p_{18}(x)$ is supposed to have 6 roots.

We have constructed p_{18} as promised.

5.2 General Theorem: If n is Even Then deg$\,(\cos(v\pi/n)) \le \phi(n)$

Theorem 5.1 *Let $n \ge 2$, n even. Let*

$$\mathrm{CO_n} = \left\{\cos\left(\frac{\mathrm{v}\pi}{\mathrm{n}}\right) : 1 \le \mathrm{v} \le \mathrm{n}-1 \quad \textit{and} \quad \gcd(\mathrm{v},\mathrm{n}) = 1\right\},$$

1. *There is a polynomial of degree $\phi(n)$ whose roots are the elements of* $\mathrm{CO_n}$.
2. *Every element of* $\mathrm{CO_n}$ *has degree $\le \phi(n)$. (This follows from Part 1.)*

Proof: We construct $\mathrm{co_n} \in \mathsf{Z}[\mathrm{x}]$ of degree $\phi(n)$ whose roots are $\mathrm{CO_n}$.
The construction is by induction on n.
Base Case: $n = 2$. Then $\mathrm{CO_n} = \{\cos(\pi/2)\} = \{0\}$. Let $\mathrm{co_2(x) = x}$. $\mathrm{co_2(x)}$ has degree 1.

Induction Hypothesis Assume n is even and $n \ge 4$. Assume that, for $2 \le n' < n$, n' even, there exists $\mathrm{co_{n'}} \in \mathsf{Z}[\mathrm{x}]$ of degree $\phi(n')$ whose roots are

$$\mathrm{CO_{n'}} = \left\{\cos\left(\frac{\mathrm{v}\pi}{\mathrm{n'}}\right) : 1 \le \mathrm{v} \le \mathrm{n'}-1 \quad \text{and} \quad \gcd(\mathrm{v},\mathrm{n'}) = 1\right\}.$$

Induction Step

To construct $\mathrm{co_n}$ we first take $T_{2n-1}(x) - x$. By Lemma 3.1.2 the roots of $T_{2n-1}(x) - x$ are

$$\left\{\cos\left(\frac{2v\pi}{2n-2}\right) : 0 \le v \le \frac{2n-2}{2}\right\} \cup \left\{\cos\left(\frac{2v\pi}{2n}\right) : 1 \le v \le \frac{2n-2}{2}\right\}$$
$$= \left\{\cos\left(\frac{v\pi}{n-1}\right) : 0 \le v \le n-1\right\} \cup \left\{\cos\left(\frac{v\pi}{n}\right) : 1 \le v \le n-1\right\}.$$

Note that $\mathrm{CO_n}$ is a subset of the roots of $T_{2n-1}(x) - x$. To remove the other roots we will divide $T_{35}(x) - x$ by some polynomials. We list sets of roots and the polynomial that has exactly that set of roots. We also include degrees for an attempt at a sanity check. For that purpose we point out that the degree of $T_{2n-1}(x) - x$ is $2n - 1$.

1. $\{\cos(0\pi/(n-1))\} = \{\cos(0)\} = \{1\}$.

   The polynomial $x - 1$ of degree 1 suffices.

2. $\{\cos((n-1)\pi/(n-1))\} = \{\cos(\pi)\} = \{-1\}$.

   The polynomial $x + 1$ of degree 1 suffices.

3. $\{\cos(k\pi/(n-1)) : 1 \le k \le n-2\}$.

   By Corollary 4.3.1, there exists $r_{n-1} \in \mathsf{Z}[x]$ of degree $n - 1$ whose roots are this set.

4. For all $2 \leq n' \leq n-2$ such that n' divides n we define:

$$CO_{n'} = \left\{ \cos\left(\frac{v\pi}{n'}\right) : 1 \leq v \leq n'-1 \ \text{ and } \ \gcd(v, n') = 1 \right\}.$$

   There are two subcases:

   (a) If n' is odd then, by Corollary 4.3.2, there is a polynomial $\mathrm{co}_{\mathrm{n'}} \in \mathsf{Z}[\mathrm{x}]$ of degree $\phi(n')$ whose roots are $CO_{n'}$.

   (b) If n' is even then, by the induction hypothesis, there is a polynomial $\mathrm{co}_{\mathrm{n'}}(\mathrm{x}) \in \mathsf{Z}[\mathrm{x}]$ of degree $\phi(n')$ whose roots are $CO_{n'}$.

For notational convenience we define two polynomials before defining co_{n}.

1. $\mathrm{ProdOdd}_n$ is the product of all $\mathrm{co}_{\mathrm{n'}}$ such that $2 \leq n' \leq n-1$, $n'|n$, and n is odd.

2. $\mathrm{ProdEven}_n$ is the product of all $\mathrm{co}_{\mathrm{n'}}$ such that $2 \leq n' \leq n-1$, $n'|n$, and n is even.

All of the roots of $T_{2n-1}(x) - x$ that are not in CO_{n} are roots of either $x-1$, $x+1$, $\mathrm{ProdOdd}_n(x)$, or $\mathrm{ProdEven}_n(x)$. Hence we take

$$\mathrm{co}_{\mathrm{n}}(\mathrm{x}) = \frac{\mathrm{T}_{2\mathrm{n}-1}(\mathrm{x}) - \mathrm{x}}{(\mathrm{x}-1)(\mathrm{x}+1)\mathrm{r}_{\mathrm{n}-1}(\mathrm{x})\mathrm{ProodOdd}_{\mathrm{n}}(\mathrm{x})\mathrm{ProdEven}_{\mathrm{n}}(\mathrm{x})}$$

$\mathrm{co}_{\mathrm{n}} \in \mathsf{Z}[\mathrm{x}]$ by Theorem 2.3. The set of roots of co_{n} is CO_{n}.

As an attempt at a sanity check we calculate the degree of co_{n} based on the degrees of the numerator and denominator in the definition of co_{n}.

To write down the degree of co_{n} we note the following:

1. $\deg(\mathrm{ProdOdd}_n)$ is the sum over all n' such that $2 \leq n' \leq n-1$, $n'|n$, and n is odd, of $\phi(n')$.

2. $\deg(\mathrm{ProdEven}_n)$ is the sum over all n' such that $2 \leq n' \leq n-1$, $n'|n$, and m is even, of $\phi(n')$.

The degree of co_{n} is

$2n-1-1-1-(n-1) - \deg(\mathrm{ProdEven}_n) - \deg(\mathrm{ProdOdd}_n) = n-2-\deg(\mathrm{ProdEven}_n) - \deg(\mathrm{ProdOdd}_n)$.

We also know that the there are exactly $\phi(n)$ roots of co_{n}. We can now view these two expressions for the degree in two ways.

1. We have shown $\phi(n) = n-2-\deg(\mathrm{ProdEven}_n) - \deg(\mathrm{ProdOdd}_n)$.

2. We would like to have an independent proof that $\phi(n) = n-2-\deg(\mathrm{ProdEven}_n) - \deg(\mathrm{ProdOdd}_n)$ as a sanity check.

We have constructed co_{n} as promised. ∎

6 Upper Bounds on deg (sin $(v\pi/n)$)

Fact 6.1 *For all* θ*,* $\sin(\theta) = \cos(\theta - \pi/2)$*.*

Theorem 6.2 *Let* n, v *be such that* $\gcd(v, n) = 1$*.*

1. *If* $n \equiv 0 \pmod 4$ *then* $\deg(\sin(v\pi/n)) \le \phi(n)$*.*
2. *If* $n \equiv 2 \pmod 4$ *then* $\deg(\sin(v\pi/n)) \le \phi(n/2)/2$*.*
3. *If* n *is odd then* $\deg(\sin(v\pi/n)) \le \phi(2n)$*.*

Proof:

$$\sin\left(\frac{v\pi}{n}\right) = \cos\left(\frac{v\pi}{n} - \frac{\pi}{2}\right) = \cos\left(\frac{(2v-n)\pi}{2n}\right).$$

Hence for all three cases we find $\deg(\cos((2v - n)\pi/2n))$ and use Theorem 4.2 or 5.1.
Part 1: $n \equiv 0 \pmod 4$. Let m be such that $n = 4m$. Then

$$\frac{(2v-n)\pi}{2n} = \frac{(2v-4m)\pi}{8m} = \frac{(v-2m)\pi}{4m}.$$

To use either Theorem 5.1 we need $\gcd(v - 2m, 4m) = 1$.
Claim: $\gcd(v - 2m, 4m) = 1$.
Proof of Claim:
$\gcd(v, 4m) = \gcd(v, n) = 1$. Hence v is odd.
Assume d divides both $v - 2m$ and $4m$. We show $d = 1$ by (a) showing that d divides both n and v, and (b) using $\gcd(v, n) = 1$.

- Since v is odd, $v - 2m$ is odd.
- Since d divides $v - 2m$, and $v - 2m$ is odd, d is odd.
- Since d divides both $4m$ and $v - 2m$, d divides $2(v - 2m) + 4m = 2v$.
- Since d is odd and v divides $2v$, d divides v.
- d divides $4m = n$.

End of Proof of Claim
Since $\gcd(v - 2m, 4m) = 1$, by Theorem 5.1:

$$\deg\left(\cos\left(\frac{(v-2m)\pi}{4m}\right)\right) = \phi(4m) = \phi(n).$$

By Fact 6.1 and the above line,

$$\deg\left(\sin\left(\frac{v\pi}{n}\right)\right) = \deg\left(\cos\left(\frac{(v-2m)\pi}{4m}\right)\right) = \phi(n).$$

Part 2: $n \equiv 2 \pmod 4$. Let m be such that $n = 4m+2$.

$$\frac{(2v-n)\pi}{2n} = \frac{(2v-4m-2)\pi}{8m+4} = \frac{(v-2m-1)\pi}{4m+2}.$$

Since $\gcd(v,n)=1$, v is odd. Let $v = 2w+1$. Then

$$\frac{(v-2m-1)\pi}{4m+2} = \frac{(2w-2m)\pi}{4m+2} = \frac{(w-m)\pi}{2m+1}$$

To use Theorem 4.2 we need $\gcd(w-m, 2m+1) = 1$.
Claim: $\gcd(w-m, 2m+1) = 1$.
Proof of Claim:

Assume d divides both $w-m$ and $2m+1 = n/2$. We show $d=1$ by (a) showing that d divides both n and v, and (b) using $\gcd(v,n)=1$.

- Since d divides $n/2$, d divides n.
- Since d divides both $w-m$ and $2m+1$, d divides $2(w-m)+2m+1 = 2w+1 = v$.

End of Proof of Claim

Since $\gcd(w-n, 2m+1) = 1$, by Theorem 4.2

$$\deg\left(\cos\left(\frac{(w-m)\pi}{2m+1}\right)\right) = \frac{\phi(2m+1)}{2} = \frac{\phi(\frac{n}{2})}{2}.$$

By Fact 6.1 and the above line,

$$\deg\left(\sin\left(\frac{v\pi}{n}\right)\right) = \deg\left(\cos\left(\frac{(w-m)\pi}{2m+1}\right)\right) = \frac{\phi(\frac{n}{2})}{2}.$$

Part 3: $n \equiv 1 \pmod 2$. Let m be such that $n = 2m+1$.

$$\frac{(2v-n)\pi}{2n} = \frac{(2v-2m-1)\pi}{4m+2}.$$

To use Theorem 5.1 we need $\gcd(2v-2m-1, 4m+2) = 1$.
Claim: $\gcd(2v-2m-1, 4m+2) = 1$.

Assume d divides both $2v-2m-1$ and $4m+2 = 2n$. We show $d=1$ by (a) showing that d divides both n and v, and (b) using $\gcd(v,n)=1$.

- Since d divides $2v-2m-1$, d is odd.
- Since d divides $4m+2=2n$, and d is odd, d divides n.

- Since d divides both $2v-2m-1$ and $4m+2$, d divides $2(2v-2m-1)+4m+2 = 4v$.
- Since d divides $4v$ and d is odd, d divides v.

End of Proof of Claim

Since $\gcd(2v-2m-1, 4m+2) = 1$, by Theorem 5.1:

$$\deg\left(\cos\left(\frac{(2v-2m-1)\pi}{4m+2}\right)\right) = \phi(4m+2) = \phi(2n).$$

By Fact 6.1 and the above line,

$$\deg\left(\sin\left(\frac{v\pi}{n}\right)\right) = \deg\left(\cos\left(\frac{(2v-2m-1)\pi}{4m+2}\right)\right) = \phi(2n).$$

∎

7 $\deg(\cos(v\pi/n))$ & $\deg(\sin(v\pi/n))$: Field Theory

7.1 Background Needed

We state well known facts from field theory and use them to prove our results. All fields are subsets of C.

Definition 7.1 Let F and E be fields. E is *a field extension of* F if

- $\mathsf{F} \subseteq \mathsf{E}$.
- The operations $+, \times$ in F are $+, \times$ in E restricted to F.

Fact 7.2

1. *If* E *is a field extension of* F *then* E *is a vector space over* F. *We denote the dimension of this vector space by* $[\mathsf{E} : \mathsf{F}]$.
2. *If* D *is a field extension of* E *and* E *is a field extension of* F *then* $[\mathsf{D} : \mathsf{F}] = [\mathsf{D} : \mathsf{E}][\mathsf{E} : \mathsf{F}]$.

Definition 7.3 Let $\mathsf{F} \subseteq \mathsf{C}$ be a field and let $\alpha \in \mathsf{C} - \mathsf{F}$.

$$\mathsf{F}(\alpha) = \left\{\frac{p(\alpha)}{q(\alpha)} : p, q \in \mathsf{F}[x] \text{ and } q(\alpha) \neq 0\right\}.$$

Definition 7.4 Let E be a field extension of F. Let $\alpha \in \mathsf{E}$. *The degree of* α over F is the smallest $d \in \mathsf{N}$ such that α is the root of a degree-d polynomial in $\mathsf{F}[x]$. We denote this by $\deg_{\mathsf{F}}(\alpha)$. If $\mathsf{F} = \mathsf{Q}$ then we just use deg which matches the definition of deg we have been using.

Fact 7.5 *$\mathsf{F}(\alpha)$ is a field extension of F and $[\mathsf{F}(\alpha) : \mathsf{F}] = \deg_{\mathsf{F}}(\alpha)$.*

Proof:

Clearly $\mathsf{F}(\alpha)$ is a field extension of F.

Let $\deg_F(\alpha) = d$.

We show that The set $\{1, \alpha, \alpha^2, \ldots, \alpha^{d-1}\}$ forms a basis for $[\mathsf{F}(\alpha) : \mathsf{F}]$.

- Every element of $\mathsf{F}(\alpha)$ is a polynomial in α with coefficients in F. Since $\deg_F(\alpha) = d$, the polynomials can be made to be of degree $\le d$.
- Let $a_0, \ldots, a_{d-1} \in \mathsf{F}$ be such that $\sum_{i=0}^{d-1} a_i \alpha^i = 0$. Since $\deg_F(\alpha) = d$, all of the a_i are 0.

▮

Note 7.6 Lets say you prove that $[\mathsf{Q}(\alpha) : \mathsf{Q}] = d$, so $\deg(\alpha) = d$. Can Fact 7.5 help *find* a polynomial of degree d that has α as a root. No. All you find out is that $\{1, \alpha, \ldots, \alpha^d\}$ is linearly dependent over Q, hence *there exists* such a polynomial. But the proof of Fact 7.5 does not say how to *find* the polynomial.

BEGINNING OF COMMENTS TO AUGUSTE

(I DO NOT KNOW IF THE NOTE ABOVE IS CORRECT.)

We just proved $[\mathsf{F}(\alpha) : F] = \deg_{\mathsf{F}}(\alpha)$ but for us for now lets just consider $[\mathsf{Q}(\alpha) : F] = \deg_{\mathsf{Q}}(\alpha)$

1) The proof is constructive in one direction: Given α we can get a basis, namely

$$\{1, \alpha, \ldots, \alpha^{d-1}\}.$$

(Note- not clear what *given* means since α is irrational.)

2) Can the following be done: Given α and d where one is told that there is a poly $p \in \mathsf{Z}[x]$ of degree d that has α as a root, find that poly?

Actually the answer is yes for a stupid way: enumerate all polys and test each one until you find one. But even this is not really right since α is irrational so this would need perfect real arithmetic.

It may be that for our case of $\cos(v\pi/n)$ this can be dealt with.

So the question is, is there a SANE algorithm.

3) In Lemma 7.10 below we prove the following:

——-

Let $1 \le v \le n-1$ be such that $\gcd(v,n) = 1$.

$[\mathsf{Q}(\cos(2\pi v/n)) : \mathsf{Q}] = \phi(n)/2$.

Hence $\deg(\cos(2\pi v/n)) = \phi(n)/2$.

————-

SO here are my questions:

From the proof of this one can one, given v, n (that is an input you CAN be given) find poly $p \in \mathsf{Z}[x]$ of degree $\phi(n)/2$ that has α as a root.

If so, then (a) is the algorithm SANE, and (b) does the algorithm need perfect arithmetic for reals?

Much like Maya's personal statement, I don't want our final paper to dwell on this point. I want to BRIEFLY talk about how the proof using Field theory can or cannot be used to find he poly, and if yes then does or does not use real arithmetic. I will then also state this as probably one of the CONS when I discuss PROS and CONS early in the paper

END OF COMMENTS TO AUGUSTE

Notation 7.7 $\zeta_n = e^{2\pi i/n}$. (ζ is the Greek letter zeta.)

Definition 7.8 Let $n \in \mathsf{N}$. α is an *nth root of unity* if $\alpha^n = 1$. α is a *primitive root of unity* if (1) $\alpha^n = 1$, and (2) for every $n' < n$, $\alpha^{n'} \ne 1$.

Fact 7.9

1. *There are n nth roots of unity: $\zeta_n^1, \ldots, \zeta_n^n$.*
2. *There are $\phi(n)$ primitive nth roots of unity: $\{\zeta_n^v \colon \gcd(v,n) = 1\}$.*
3. *If α is a primitive nth root of unity then $\deg(\alpha) = \phi(n)$.*

7.2 deg (cos ($v\pi/n$)) Via Field Theory

Lemma 7.10 *Let $1 \le v \le n-1$ be such that* $\gcd(v,n) = 1$.

1. $[\mathsf{Q}(\zeta_n^v) : \mathsf{Q}] = \phi(n)$.
2. *If $n \ge 3$ then* $[\mathsf{Q}(\zeta_n^v) : \mathsf{Q}(\cos(2\pi v/n))] = 2$.
3. $[\mathsf{Q}(\cos(2\pi v/n)) : \mathsf{Q}] = \phi(n)/2$.
4. $\deg(\cos(2\pi v/n)) = \phi(n)/2$.

Proof:
1) $[\mathsf{Q}(\zeta_n^v):\mathsf{Q}]=\phi(n)$ follows from Fact 7.5 and Fact 7.9.3.

2) $[\mathsf{Q}(\zeta_n^v):\mathsf{Q}(\cos(2\pi v/n))]=\deg_{\mathsf{Q}(\cos(2\pi v/n))}(\zeta_n^v)$. Hence we need to show $\deg_{\mathsf{Q}(\cos(2\pi v/n))}(\zeta_n^v)=2$.

a) We show $\deg_{\mathsf{Q}(\cos(2\pi v/n))}(\zeta_n^v)\le 2$.
We derive a quadratic polynomial with coefficients in $\mathsf{Q}(\cos(2\pi v/n))$ that has $e^{2\pi vi/n}$ as a root. Then $\deg_{\mathsf{Q}(\cos(2\pi v/n))}(\zeta_n^v)\le 2$.
Let $\beta=e^{2\pi vi/n}$. Recall that

$$\cos(x)=\frac{e^{ix}+e^{-ix}}{2}.$$

Hence

$$\cos\left(\frac{2\pi v}{n}\right)=\frac{\beta+\frac{1}{\beta}}{2}.$$

Hence we need a polynomial with coefficients in $\mathsf{Q}(\beta+1/\beta)$ that has β as a root. The polynomial

$$x^2-(\beta+1/\beta)x+1=0$$

has β as a root. Hence we take the polynomial

$$x^2-2\cos(2v\pi/n)x+1.$$

b) We show $\deg_{\mathsf{Q}(\cos(2\pi v/n))}(\zeta_n^v)\ge 2$.
Assume, by way of contradiction, that ζ_n^v is the root of a linear polynomial with coefficients in $\mathsf{Q}(\cos(2\pi v/n))$. Then $\zeta_n^v\in\mathsf{Q}(\cos(2\pi v/n))$ and hence $\zeta_n^v\in\mathsf{R}$. Since $n\ge 3$, $\zeta_n^v\in\mathsf{C}-\mathsf{R}$. This is a contradiction.

3) By Fact 7.2.2

$$[\mathsf{Q}(\zeta_n^v):\mathsf{Q}]=[\mathsf{Q}(\zeta_n^v):\mathsf{Q}(\cos(2\pi v/n))][\mathsf{Q}(\cos(2\pi v/n)):\mathsf{Q}]$$

By Part 1 and Part 2 we have

$$\phi(n)=2[\mathsf{Q}(\cos(2\pi v/n)):\mathsf{Q}].$$

Hence $[\mathsf{Q}(\cos(2\pi v/n)):\mathsf{Q}]=\phi(n)/2$.

4) By Part 3 $[\mathsf{Q}(\cos(2\pi v/n)):\mathsf{Q}]=\phi(n)/2$. By Fact 7.5

$$\deg(\cos(2\pi v/n)))=[\mathsf{Q}(\cos(2\pi v/n)):\mathsf{Q}]=\phi(n)/2.$$

∎

Theorem 7.11 *Let* $1 \le v \le n$ *such that* $\gcd(v,n) = 1$.

1. *If* n *is odd then* $\deg(\cos(v\pi/n)) = \phi(n)/2$.
2. *If* n *is even then* $\deg(\cos(v\pi/n)) = \phi(n)$.

Proof:
1) n **is odd.** There are two cases

Case 0: v is even. Then $v = 2v'$. Hence $\deg(\cos(v\pi/n)) = \deg(\cos(2v'\pi/n))$.

Since $\gcd(v,n) = 1$, $\gcd(v',n) = 1$. Hence, by Lemma 7.10.4, $\deg(\cos(2v'\pi/n)) = \phi(n)/2$, so $\deg(\cos(v\pi/n)) = \phi(n)/2$.

Case 1: v is odd. Note that $\deg(\cos(v\pi/n)) = \deg(\cos(2v\pi/2n))$.

Since v is odd and $\gcd(v,n) = 1$, $\gcd(v,2n) = 1$. Hence, by Lemma 7.10.4, $\deg(\cos(2v\pi/2n)) = \phi(2n)/2$. Since n is odd, $\phi(2n) = \phi(n)$ so $\deg(\cos(v\pi/n)) = \phi(n)/2$.

2) n **is even.** Note that $\deg(\cos(v\pi/n)) = \deg(\cos(2v\pi/2n))$.

Since n is even and $\gcd(v,n) = 1$, $\gcd(v,2n) = 1$. Hence, by Lemma 7.10.4, $\deg(\cos(2v\pi/2n)) = \phi(2n)/2$, Since n is even, $\phi(2n) = 2\phi(n)$ so $\deg(\cos(v\pi/n)) = \phi(n)$. ∎

7.3 deg (sin $(v\pi/n)$) Via Field Theory

BILL- WILL PROB RE DO THIS ENTIRE SECTION, ON FIELD THEORY PROOF FOR SINE. LATER

Lemma 7.12 *Let* $1 \le v \le n-1$ *be such that* $\gcd(v,n) = 1$. *Let* $\zeta_n = e^{2\pi i/n}$.

1. $[Q(\zeta_n) : Q] = \phi(n)$.
2. $[\mathsf{Q}(\sin(2v\pi/n), i)) : \mathsf{Q}(\sin(2v\pi/n))] = 2$.

Proof:
1) By Fact 7.5 $[Q(\zeta_n) : Q] = \deg(\zeta_n)$. By Fact 7.9, $\deg(\zeta_n) = \phi(n)$. Hence $[Q(\zeta_n) : Q] = \phi(n)$.

2) By Fact 7.5,

$$[\mathsf{Q}(\sin(2v\pi/n), i)) : \mathsf{Q}(\sin(2v\pi/n))] = \deg_{\mathsf{Q}(\sin(2v\pi/n))}(i)].$$

Since $i \notin \mathsf{Q}(\sin(2v\pi/n))$,

$$\deg_{\mathsf{Q}(\sin(2v\pi/n))}(i) \ge 2.$$

Since i is a root of x^2+1,

$$\deg_{\mathsf{Q}(\sin(2v\pi/n))}(i) \le 2.$$

Hence

$$\deg_{\mathsf{Q}(\sin(2v\pi/n))}(i) = 2.$$

∎

Lemma 7.13 *Let $1 \le v \le n-1$ be such that $n \equiv 0 \pmod 4$ and $\gcd(v,n) = 1$. Let $\zeta_n = e^{2\pi i/n}$.*

1. *If n is a power of 2 then $[\mathsf{Q}(\zeta_n) : \mathsf{Q}(\sin(2v\pi/n, i))] = 1$.*
2. *If n is not a power of 2 then $[\mathsf{Q}(\zeta_n) : \mathsf{Q}(\sin(2v\pi/n, i))] = 2$.*

Proof:
$\blacksquare$

Lemma 7.14 *Let $1 \le v \le n-1$ be such that $n \equiv 0 \pmod 4$ and $\gcd(v,n) = 1$. Let $\zeta_n = e^{2\pi i/n}$.*

1. *If n is a power of 2 then $\deg(\sin(2v\pi/n) = \phi(n)/2$.*
2. *If n is not a power of 2 then $\deg(\sin(2v\pi/n) = \phi(n)/4$.*

Proof:
1)
$[\mathsf{Q}(\zeta_n) : \mathsf{Q}] = [\mathsf{Q}(\zeta_n) : \mathsf{Q}(\sin(2v\pi/n), i))][\mathsf{Q}(\sin(2v\pi/n), i)) : \mathsf{Q}(2v\pi/n)][\mathsf{Q}(2v\pi/n) : \mathsf{Q}]$.

- By Lemma 7.12.1$[\mathsf{Q}(\zeta_n) : \mathsf{Q}] = \phi(n)$.
- By Lemma 7.15.1 $[\mathsf{Q}(\zeta_n) : \mathsf{Q}(\sin(2v\pi/n, i))] = 1$.

  BILL- ABOVE LEMMA IS IN THE FUTURE. FIX IF NEEDED- THIS SECTION WILL PROB BE REDONE
- By Lemma 7.13.2 $[\mathsf{Q}(\sin(2v\pi/n, i)) : \mathsf{Q}(2v\pi/n)][\mathsf{Q}(2v\pi/n) : \mathsf{Q}] = 2$.

Hence we have

$$\phi(n) = 1 \times 2 \times [\mathsf{Q}(2v\pi/n) : \mathsf{Q}]$$

So

$$[\mathsf{Q}(2v\pi/n) : \mathsf{Q}] = \phi(n)/2.$$

2)
$[\mathsf{Q}(\zeta_n) : \mathsf{Q}] = [\mathsf{Q}(\zeta_n) : \mathsf{Q}(\sin(2v\pi/n, i))][\mathsf{Q}(\sin(2v\pi/n, i)) : \mathsf{Q}(\sin(2v\pi/n))][\mathsf{Q}(\sin(2v\pi/n)) : \mathsf{Q}]$.

- By Lemma 7.12.1$[\mathsf{Q}(\zeta_n) : \mathsf{Q}] = \phi(n)$.
- By Lemma 7.15.2 $[\mathsf{Q}(\zeta_n) : \mathsf{Q}(\sin(2v\pi/n, i))] = 2$.

- By Lemma 7.13.2 $[\mathsf{Q}(\sin(2v\pi/n, i) : \mathsf{Q}(\sin(2v\pi/n))][\mathsf{Q}(\sin(2v\pi/n)) : \mathsf{Q}] = 2$.

  Hence we have

$$\phi(n) = 2 \times 2 \times [\mathsf{Q}(\sin(2v\pi/n)) : \mathsf{Q}]$$

  So

$$[\mathsf{Q}(\sin(2v\pi/n)) : \mathsf{Q}] = \frac{1}{4}\phi(n).$$

∎

Lemma 7.15 *Let* $0 \le v \le n$ *such that* $\gcd(v, n) = 1$.

1. *If* n *is even then*
   (a) *If* n *is a power of 2 then* $\deg(\sin(v\pi/n) = \phi(n)$.
   (b) *If* n *is not a power of 2 then* $\deg(\sin(v\pi/n) = \phi(n)/2$.
2. *If* n *is odd then BILL FILL IN. THIS SECTION WILL PROB BE REDONE*

Proof:
1) Since $n \equiv 0 \pmod 4$, $2n \equiv 0 \pmod 4$. Since $\gcd(v, n) = 1$, v is odd, so $\gcd(n, 2n) = 1$. Note that

$$\sin(v\pi/n) = \sin(2v\pi/2n).$$

If n is a power of 2 then $2n$ is a power of 2 so, by Lemma 7.15.1,

$$\deg(\sin(v\pi/n)) = \frac{1}{2}\phi(2n).$$

a) Let $n = 2^k$. Then

$$\frac{1}{2}\phi(2n) = \frac{1}{2}\phi(2^{k+1}) = \frac{1}{2}2^k = 2^{k-1} = \phi(n).$$

If n is not a power of 2 then $2n$ is not a power of 2 so, by Lemma 7.15.2.

$$\deg(\sin(v\pi/n)) = \frac{1}{4}\phi(2n).$$

b) Let $n = 2^k m$ where m is odd. Then

$$\frac{1}{4}\phi(2n) = \frac{1}{4}\phi(2^{k+1}m) = \frac{1}{4}2^k\phi(m) = 2^{k-2}\phi(m)$$

$$= \frac{1}{2}2^{k-1}\phi(m) = \frac{1}{2}\phi(2^k)\phi(m) = \frac{1}{2}\phi(2^k m) = \frac{1}{2}\phi(n).$$

2) $\sin(v\pi/n) = \sin(2 \times 2v\pi/4n)$.

∎

A Proof That $\cos(nx) = T_n(\cos(x))$

We prove Theorem 2.2, which we restate here:

Theorem A.1 *Let* $n \ge 1$. *Let*

$$T_n(x) = \sum_{k=0}^{\lfloor n/2 \rfloor} \binom{n}{2k} (x^2-1)^k x^{n-2k}.$$

Then $\cos(nx) = T_n(\cos(x))$.

The following proof is an expanded version of an anonymous post on math.stackexchange so it is probably folklore. Here is the URL:

`https://math.stackexchange.com/questions/125774/how-to-expand-cos-nx-with-cos-x`

Proof:

As usual, $i = \sqrt{-1}$.

We view e^{inx} in two ways.

WAY ONE

$$e^{inx} = \sum_{j=0}^{\infty} \frac{(inx)^j}{j!} = \sum_{j\equiv 0 \bmod 2}^{\infty} \frac{i^j(nx)^j}{j!} + \sum_{j\equiv 1 \bmod 2}^{\infty} \frac{i^j(nx)^j}{j!}$$

$$= \sum_{j\equiv 0 \bmod 2}^{\infty} \frac{(-1)^{j/2}(nx)^j}{j!} + \sum_{j\equiv 1 \bmod 2}^{\infty} \frac{(-1)^{((j-1)/2)}(nx)^j}{j!} i$$

$$= \sum_{k=0}^{\infty} \frac{(-1)^k(nx)^{2k}}{(2k)!} + \sum_{j\equiv 1 \bmod 2}^{\infty} \frac{(-1)^{((j-1)/2)}(nx)^j}{j!} i$$

$$\cos(nx) + \sum_{j\equiv 1 \bmod 2}^{\infty} \frac{(-1)^{((j-1)/2)}(nx)^j}{j!} i.$$

So the real part of e^{inx} is $\cos(nx)$.

WAY TWO

$$e^{inx} = (e^{ix})^n = (\cos(x) + i\sin(x))^n = \sum_{j=0}^{n} \binom{n}{j} i^j \sin^j(x)\cos^{n-j}(x).$$

$$= \sum_{j\equiv 0 \bmod 2, j\le n}^{n} \binom{n}{j} i^j \sin^j(x)\cos^{n-j}(x) + \sum_{j\equiv 1 \bmod 2, j\le n}^{n} \binom{n}{j} i^j \sin^j(x)\cos^{n-j}(x) +$$

$$= \sum_{j\equiv 0 \bmod 2, j\le n}^{n} \binom{n}{j}(-1)^{j/2}\sin^j(x)\cos^{n-j}(x) + \sum_{j\equiv 1 \bmod 2, j\le n}^{n} \binom{n}{j}(-1)^{(j-1)/2}\sin^j(x)\cos^{n-j}(x)i$$

$$= \sum_{k=0}^{\lfloor n/2\rfloor} \binom{n}{2k}(-1)^k\sin^{2k}(x)\cos^{n-2k}(x) + \sum_{j\equiv 1 \bmod 2, j\le n} \binom{n}{j}1^{(j-1)/2}\sin^j(x)\cos^{n-j}(x)i$$

$$= \sum_{k=0}^{\lfloor n/2\rfloor} \binom{n}{2k}(-1)^k(\sin^2(x))^k\cos^{n-2k}(x) + \sum_{j\equiv 1 \bmod 2, j\le n} \binom{n}{j}1^{(j-1)/2}\sin^j(x)\cos^{n-j}(x)i$$

$$= \sum_{k=0}^{\lfloor n/2\rfloor} \binom{n}{2k}(-1)^k(1-\cos^2(x))^k\cos^{n-2k}(x) + \sum_{j\equiv 1 \bmod 2, j\le n} \binom{n}{j}1^{(j-1)/2}\sin^j(x)\cos^{n-j}(x)i$$

$$= \sum_{k=0}^{\lfloor n/2\rfloor} \binom{n}{2k}(\cos^2(x)-1)^k\cos^{n-2k}(x) + \sum_{j\equiv 1 \bmod 2, j\le n} \binom{n}{j}1^{(j-1)/2}\sin^j(x)\cos^{n-j}(x)i$$

WAY ONE gives that the real part of e^{inx} is $\cos(nx)$. WAY TWO gives that the real part of e^{inx} is $\sum_{k=0}(x)^{\lfloor n/2\rfloor}\binom{n}{2k}(\cos^2(x)-1)^k\cos^{n-2k}(x)$

By equating the real part of WAY ONE and the real part of WAY TWO we get the theorem sought. ▮

B Lemmas on Polynomial Divisibility in $\mathsf{Z}[x]$

Definition B.1 A polynomial in $\mathsf{Z}[x]$ is *primitive* if the gcd of its coefficients is 1.

Lemma B.2

1. *For all $n \ge 1$, the coefficient of x^n in $T_n(x)$ is 2^{n-1}.*
2. *For all $n \ge 1$, n is even, T_n has constant term $(-1)^{n/2}$.*
3. *For all $n \ge 1$, n is odd, T_n has linear term $(-1)^{(n-1/2}nx$.*
4. *For all $n \ge 1$, T_n is primitive.*

Proof: Recall that

$$T_n(x) = \sum_{k=0}^{\lfloor n/2 \rfloor} \binom{n}{2k} (x^2-1)^k x^{n-2k}.$$

1) The part of the sum that contains x^n is

$$\sum_{k=0}^{\lfloor n/2 \rfloor} \binom{n}{2k} (x^2)^k x^{n-2k} = \sum_{k=0}^{\lfloor n/2 \rfloor} \binom{n}{2k} x^n = 2^{n-1} x^n.$$

Hence the coefficient of x^n is 2^{n-1}.

2) The constant term is part of the summand when $k = n/2$. This summand is

$$\binom{n}{n} (x^2-1)^{n/2} x^{n-2(n/2)} = (x^2-1)^{n/2}$$

Hence the constant term is $(-1)^{n/2}$.

3) The linear term is part of the summand when $k = (n-1)/2$. This summand is

$$\binom{n}{n-1} (x^2-1)^{(n-1)/2} x = n(x^2-1)^{(n-1)/2} x$$

Hence the linear term is $(-1)^{(n-1)/2} nx$.

4) There are two cases.

- If n is even then the constant term is $(-1)^{n/2}$, hence the gcd of all the coefficients is 1. Hence T_n is primitive.
- If n is odd then the coefficient of x is $(-1)^{(n-1/2} nx$ which is odd. The coefficient of x^n is 2^{n-1}. Since the gcd of an odd number and a power of 2 is 1, the gcd of all the coefficients is 1. Hence T_n is primitive.

∎

Lemma B.3 *Let $T, p \in \mathsf{Z}[x]$ such that T is primitive.*

1. *If p divides T in $\mathsf{C}[x]$ then $T/p \in \mathsf{Q}[x]$.*
2. *If p divides T in $\mathsf{Q}[x]$ then $T/p \in \mathsf{Z}[x]$.*
3. *If p divides T in $\mathsf{C}[x]$ then $T/p \in \mathsf{Z}[x]$. (This follows from parts 1 and 2.)*

Proof:
1) (This part does not use that T is primitive.) Let $T/p = q$. Since $T, p \in \mathsf{Z}[x]$ we know that $q(0), q(1), \ldots$ are all in Q.

Let $q(x) = a_n x^n + \cdots + a_0$. Then
$(a_n, \cdots, a_0) \cdot (0, \cdots, 0, 1) = q(0)$
$(a_n, \cdots, a_0) \cdot (1, \cdots, 1, 1) = q(1)$
$(a_n, \cdots, a_0) \cdot (2^n, \cdots, 2^1, 2^0) = q(2)$
$(a_n, \cdots, a_0) \cdot (3^n, \cdots, 3^1, 3^0) = q(3)$
$\vdots \qquad \vdots$
$(a_n, \cdots, a_0) \cdot (n^n, \cdots n^0) = q(n)$
$(a_0, \ldots, a_n)$ is the solution to $n+1$ equations over Q. Hence $a_0, \ldots, a_n \in \mathsf{Q}$.

2) Let $T/p = q$ where $q \in \mathsf{Q}[x]$. Then $T = pq$. Since $p \in \mathsf{Z}[x]$ there exists $a \in \mathsf{Z}$ such that $p = ap^\dagger$ where $p^\dagger \in \mathsf{Z}[x]$ is primitive. Since $q \in \mathsf{Z}[x]$ there exists $b \in \mathsf{Z}$ such that $q = bq^\dagger$ where $q^\dagger \in \mathsf{Z}[x]$ is primitive. Hence we have
$T = abp^\dagger q^\dagger$
Since T is primitive $ab = 1$. Hence $b \in \{1, -1\}$ so $q \in \mathsf{Z}[x]$. ∎

Theorem B.4 *Let $n \geq 1$. Let $p \in \mathsf{Z}[x]$. If the set of roots of p are a subset of the set of roots of T_n then $\frac{T(x)}{p(x)} \in \mathsf{Z}[x]$.*

Proof: By Lemma B.2 T_n is primitive. Since the set of roots of p is a subset of the set of roots of T_n, p divides T_n. By Lemma B.3 $\frac{T(x)}{p(x)} \in \mathsf{Z}[x]$. ∎

C The First 39 Chebyshev Polynomials

In Section D we will list out the polynomials that have $\cos(v\pi/n)$ as roots for $n = 1, \ldots, 21$, $1 \leq v \leq n-1$, $\gcd(v, n) = 1$. For the odd n we need T_{n-1}. Hence we need $T_0, T_2, \ldots, T_{20}$. For the even n we need T_{2n-1}. Hence we need $T_1, T_3, \ldots, T_{39}$. In this section we list out $T_1, \ldots, T_{39}$.

1. $T_1(x) = x$
2. $T_2(x) = 2x^2 - 1$
3. $T_3(x) = 4x^3 - 3x$
4. $T_4(x) = 8x^4 - 8x^2 + 1$
5. $T_5(x) = 16x^5 - 20x^3 + 5x$
6. $T_6(x) = 32x^6 - 48x^4 + 18x^2 - 1$

7. $T_7(x) = 64x^7 - 112x^5 + 56x^3 - 7x$

8. $T_8(x) = 128x^8 - 256x^6 + 160x^4 - 32x^2 + 1$

9. $T_9(x) = 256x^9 - 576x^7 + 432x^5 - 120x^3 + 9x$

10. $T_{10}(x) = 512x^{10} - 1280x^8 + 1120x^6 - 400x^4 + 50x^2 - 1$

11. $T_{11}(x) = 1024x^{11} - 2816x^9 + 2816x^7 - 1232x^5 + 220x^3 - 11x$

12. $T_{12}(x) = 2048x^{12} - 6144x^{10} + 6912x^8 - 3584x^6 + 840x^4 - 72x^2 + 1$

13. $T_{13}(x) = 4096x^{13} - 13312x^{11} + 16640x^9 - 9984x^7 + 2912x^5 - 364x^3 + 13x$

14. $T_{14}(x) = 8192x^{14} - 28672x^{12} + 39424x^{10} - 26880x^8 + 9408x^6 - 1568x^4 + 98x^2 - 1$

15. $T_{15}(x) = 16384x^{15} - 61440x^{13} + 92160x^{11} - 70400x^9 + 28800x^7 - 6048x^5 + 560x^3 - 15x$

16. $T_{16}(x) = 32768x^{16} - 131072x^{14} + 212992x^{12} - 180224x^{10}$
$+84480x^8 - 21504x^6 + 2688x^4 - 128x^2 + 1$

17. $T_{17}(x) = 65536x^{17} - 278528x^{15} + 487424x^{13} - 452608x^{11}$
$+239360x^9 - 71808x^7 + 11424x^5 - 816x^3 + 17x$

18. $T_{18}(x) = 131072x^{18} - 589824x^{16} + 1105920x^{14} - 1118208x^{12}$
$+658944x^{10} - 228096x^8 + 44352x^6 - 4320x^4 + 162x^2 - 1$

19. $T_{19}(x) = 262144x^{19} - 1245184x^{17} + 2490368x^{15} - 2723840x^{13}$
$+1770496x^{11} - 695552x^9 + 160512x^7 - 20064x^5 + 1140x^3 - 19x$

20. $T_{20}(x) = 524288x^{20} - 2621440x^{18} + 5570560x^{16} - 6553600x^{14}$
$+4659200x^{12} - 2050048x^{10} + 549120x^8 - 84480x^6 + 6600x^4 - 200x^2 + 1$

21. $T_{21}(x) = 1048576x^{21} - 5505024x^{19} + 12386304x^{17} - 15597568x^{15}$
$+12042240x^{13} - 5870592x^{11} + 1793792x^9 - 329472x^7 + 33264x^5 - 1540x^3 + 21x$

22. $T_{22}(x) = 2097152x^{22}(x) - 11534336x^{20} + 27394048x^{18} - 36765696x^{16}$
$+30638080x^{14} - 16400384x^{12} + 5637632x^{10} - 1208064x^8$
$+151008x^6 - 9680x^4 + 242x^2 - 1$

23. $T_{23}(x) = 4194304x^{23} - 24117248x^{21} + 60293120x^{19} - 85917696x^{17}$
$+76873728x^{15} - 44843008x^{13} + 17145856x^{11} - 4209920x^9$
$+631488x^7 - 52624x^5 + 2024x^3 - 23x$

24. $T_{24}(x) = 8388608x^{24} - 50331648x^{22} + 132120576x^{20} - 199229440x^{18}$
$+190513152x^{16} - 120324096x^{14} + 50692096x^{12} - 14057472x^{10}$
$+2471040x^{8} - 256256x^{6} + 13728x^{4} - 288x^{2} + 1$

25. $T_{25}(x) = 16777216x^{25} - 104857600x^{23} + 288358400x^{21} - 458752000x^{19}$
$+466944000x^{17} - 317521920x^{15} + 146227200x^{13} - 45260800x^{11}$
$+9152000x^{9} - 1144000x^{7} + 80080x^{5} - 2600x^{3} + 25x$

26. $T_{26}(x) = 33554432x^{26} - 218103808x^{24} + 627048448x^{22} - 1049624576x^{20}$
$+1133117440x^{18} - 825556992x^{16} + 412778496x^{14} - 141213696x^{12}$
$+32361472x^{10} - 4759040x^{8} + 416416x^{6} - 18928x^{4} + 338x^{2} - 1$

27. $T_{27}(x) = 67108864x^{27} - 452984832x^{25} + 1358954496x^{23} - 2387607552x^{21}$
$+2724986880x^{19} - 2118057984x^{17} + 1143078912x^{15} - 428654592x^{13}$
$+109983744x^{11} - 18670080x^{9} + 1976832x^{7} - 117936x^{5} + 3276x^{3} - 27x$

28. $T_{28}(x) = 134217728x^{28} - 939524096x^{26} + 2936012800x^{24} - 5402263552x^{22}$
$+6499598336x^{20} - 5369233408x^{18} + 3111714816x^{16} - 1270087680x^{14}$
$+361181184x^{12} - 69701632x^{10} + 8712704x^{8} - 652288x^{6} + 25480x^{4} - 392x^{2} + 1$

29. $T_{29}(x) = 268435456x^{29} - 1946157056x^{27} + 6325010432x^{25} - 12163481600x^{23}$
$+15386804224x^{21} - 13463453696x^{19} + 8341487616x^{17} - 3683254272x^{15}$
$+1151016960x^{13} - 249387008x^{11} + 36095488x^{9} - 3281408x^{7}$
$+168896x^{5} - 4060x^{3} + 29x$

30. $T_{30}(x) = 536870912x^{30} - 4026531840x^{28} + 13589544960x^{26} - 27262976000x^{24}$
$+36175872000x^{22} - 33426505728x^{20} + 22052208640x^{18} - 10478223360x^{16}$
$+3572121600x^{14} - 859955200x^{12} + 141892608x^{10} - 15275520x^{8}$
$+990080x^{6} - 33600x^{4} + 450x^{2} - 1$

31. $T_{31}(x) = 1073741824x^{31} - 8321499136x^{29} + 29125246976x^{27} - 60850962432x^{25}$
$+84515225600x^{23} - 82239815680x^{21} + 57567870976x^{19} - 29297934336x^{17}$
$+10827497472x^{15} - 2870927360x^{13} + 533172224x^{11} - 66646528x^{9}$
$+5261568x^{7} - 236096x^{5} + 4960x^{3} - 31x$

32. $T_{32}(x) = 2147483648x^{32} - 17179869184x^{30} + 62277025792x^{28} - 135291469824x^{26}$
$+196293427200x^{24} - 200655503360x^{22} + 148562247680x^{20} - 80648077312x^{18}$
$+32133218304x^{16} - 9313976320x^{14} + 1926299648x^{12} - 275185664x^{10}$
$+25798656x^{8} - 1462272x^{6} + 43520x^{4} - 512x^{2} + 1$

33. $T_{33}(x) = 4294967296x^{33} - 35433480192x^{31} + 132875550720x^{29} - 299708186624x^{27}$
$+453437816832x^{25} - 485826232320x^{23} + 379364311040x^{21} - 218864025600x^{19}$
$+93564370944x^{17} - 29455450112x^{15} + 6723526656x^{13} - 1083543552x^{11}$
$+118243840x^{9} - 8186112x^{7} + 323136x^{5} - 5984x^{3} + 33x$

34. $T_{34}(x) = 8589934592x^{34} - 73014444032x^{32} + 282930970624x^{30} - 661693399040x^{28}$
$+1042167103488x^{26} - 1167945891840x^{24} + 959384125440x^{22} - 586290298880x^{20}$
$+267776819200x^{18} - 91044118528x^{16} + 22761029632x^{14} - 4093386752x^{12}$
$+511673344x^{10} - 42170880x^{8} + 2108544x^{6} - 55488x^{4} + 578x^{2} - 1$

35. $T_{35}(x) = 17179869184x^{35} - 150323855360x^{33} + 601295421440x^{31} - 1456262348800x^{29}$
$+2384042393600x^{27} - 2789329600512x^{25} + 2404594483200x^{23} - 1551944908800x^{21}$
$+754417664000x^{19} - 275652608000x^{17} + 74977509376x^{15} - 14910300160x^{13}$
$+2106890240x^{11} - 202585600x^{9} + 12403200x^{7} - 434112x^{5} + 7140x^{3} - 35x$

36. $T_{36}(x) = 34359738368x^{36} - 309237645312x^{34} + 1275605286912x^{32} - 3195455668224x^{30}$
$+5429778186240x^{28} - 6620826304512x^{26} + 5977134858240x^{24} - 4063273943040x^{22}$
$+2095125626880x^{20} - 819082035200x^{18} + 240999137280x^{16} - 52581629952x^{14}$
$+8307167232x^{12} - 916844544x^{10} + 66977280x^{8} - 2976768x^{6} + 69768x^{4} - 648x^{2} + 1$

37. $T_{37}(x) = 68719476736x^{37} - 635655159808x^{35} + 2701534429184x^{33} - 6992206757888x^{31}$
$+12315818721280x^{29} - 15625695002624x^{27} + 14743599316992x^{25} - 10531142369280x^{23}$
$+5742196162560x^{21} - 2392581734400x^{19} + 757650882560x^{17} - 180140769280x^{15}$
$+31524634624x^{13} - 3940579328x^{11} + 336540160x^{9} - 18356736x^{7}$
$+573648x^{5} - 8436x^{3} + 37x$

38. $T_{38}(x) = 137438953472x^{38} - 1305670057984x^{36} + 5712306503680x^{34} - 15260018802688x^{32}$
$+27827093110784x^{30} - 36681168191488x^{28} + 36108024938496x^{26} - 27039419596800x^{24}$
$+15547666268160x^{22} - 6880289095680x^{20} + 2334383800320x^{18} - 601280675840x^{16}$
$+115630899200x^{14} - 16188325888x^{12} + 1589924864x^{10} - 103690752x^{8}$
$+4124064x^{6} - 86640x^{4} + 722x^{2} - 1$

39. $T_{39}(x) = 274877906944x^{39} - 2680059592704x^{37} + 12060268167168x^{35} - 33221572034560x^{33}$
$+62646392979456x^{31} - 85678155104256x^{29} + 87841744879616x^{27} - 68822438510592x^{25}$
$+41626474905600x^{23} - 19502774353920x^{21} + 7061349335040x^{19} - 1960212234240x^{17}$
$+411402567680x^{15} - 63901286400x^{13} + 7120429056x^{11} - 543921664x^{9}$
$+26604864x^{7} - 746928x^{5} + 9880x^{3} - 39x$

D Table of Polynomials

In the first column, if we have a number like $\pi/4$ we mean $\cos(\pi/4)$.

Roots	Polynomial
$\pi/2$	$x-1$
$\pi/3$	$-2x+1$
$2\pi/3$	$2x+1$
$\pi/4, 3\pi/4$	$2x^2-1$
$\pi/5, 3\pi/5$	$4x^2-2x-1$
$2\pi/5, 4\pi/5$	$4x^2+2x-1$
$\pi/6, 5\pi/6$	$4x^2-3$
$\pi/7, 3\pi/7, 5\pi/7$	$-8x^3+4x^2+4x-1$
$2\pi/7, 4\pi/7, 6\pi/7$	$8x^3+4x^2-4x-1$
$\pi/8, 3\pi/8, 5\pi/8, 7\pi/8$	$8x^4-8x^2+1$
$\pi/9, 5\pi/9, 7\pi/9,$	$-8x^3+6x+1$
$2\pi/9, 4\pi/9, 8\pi/9$	$8x^3-6x+1$
$\pi/10, 3\pi/10, 7\pi/10, 9\pi/10$	$16x^4-20x^2+5$
$\pi/11, 3\pi/11, 5\pi/11, 7\pi/11, 9\pi/11$	$-32x^5+16x^4+32x^3-12x^2-6x+1$
$2\pi/11, 4\pi/11, 6\pi/11, 8\pi/11, 10\pi/11$	$32x^5+16x^4-32x^3-12x^2+6x+1$
$\pi/12, 5\pi/12, 7\pi/12, 11\pi/12$	$16x^4-16x^2+1$
$\pi/13, 3\pi/13, 5\pi/13, 7\pi/13, 9\pi/13, 11\pi/13$	$64x^6-32x^5-80x^4+32x^3+24x^2-6x-1$
$2\pi/13, 4\pi/13, 6\pi/13, 8\pi/13, 10\pi/13, 12\pi/13$	$64x^6+32x^5-80x^4-32x^3+24x^2+6x-1$
$\pi/14, 3\pi/14, 5\pi/14, 9\pi/14, 11\pi/14, 13\pi/14$	$64x^6+112x^4-56x^2+7$
$\pi/15, 7\pi/15, 11\pi/15, 13\pi/15$	$16x^4+8x^3-16x^2-8x+1$
$2\pi/15, 4\pi/15, 8\pi/15, 14\pi/15$	$16x^4-8x^3-16x^2+8x+1$
$\pi/16, 3\pi/16, 5\pi/16, 7\pi/16, 9\pi/16\ldots$	$128x^8-256x^6+160x^4-32x^2+1$
$\pi/17, 3\pi/17, 5\pi/17, \ldots$	$256x^8-128x^7-448x^6+192x^5$ $+240x^4-80x^3-40x^2+8x+1$
$2\pi/17, 4\pi/17, 6\pi/17, 14\pi\ldots$	$256x^8+128x^7-448x^6-192x^5$ $+240x^4+80x^3-40x^2-8x+1$
$\pi/18, 3\pi/18, 5\pi/18, 7\pi/18, 9\pi/18\ldots$	$-64x^6+96x^4-36x^2+3$
$\pi/19, 3\pi/19, 5\pi/19, \ldots$	$-512x^9+256x^8+1024x^7-448x^6-672x^5$ $+240x^4+160x^3-40x^2-10x+1$
$2\pi/19, 4\pi/19, 6\pi/19, 8\pi/19\ldots$	$512x^9+256x^8-1024x^7-448x^6+672x^5$ $+240x^4-160x^3-40x^2+10x+1$
$\pi/20, 3\pi/20, 5\pi/20, 7\pi/20, 9\pi/20\ldots$	$256x^8-512x^6+304x^4-48x^2+1$
$\pi/21, 5\pi/21, 7\pi/21, \ldots$	$64x^6+32x^5-96x^4-48x^3+32x^2+16x+1$
$2\pi/21, 4\pi/21, 6\pi/21, 8\pi/21\ldots$	$64x^6-32x^5-96x^4+48x^3+32x^2-16x+1$

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