**Auction Theory**
**Exposition by William Gasarch**

# 1 Introduction

One way to divide a discrete good is to have an auction. In this exposition we only discussed 1-round, sealed bid auctions. So everyone writes down a bid, and after they are revealed, the higher bid gets the item. Sounds simple, but the question arises as to how much you should bid.

There are two kinds of auctions we will consider:

1. First price auction: After the bids are revealed the winner pays what he bid for the item, which is the first highest bid for the item.

2. Second price auction: After the bids are revealed the winner pays the second highest bid for the item.

We also assume that if there is a TIE then no sale takes place. There are far more realistic tie breaking mechanisms one could consider, but this one makes the math simpler and the ideas still shine through.

# 2 First Price Auction

The WINNER of the bid pays what they bid.

Lets say the bids MUST be in the set

$$B = \{0, 1, 2, 3, 4, 5, 6, 7, 8\}.$$

Lets say the item is worth 7 to Alice.

What should she bid to optimize her expected profit?

## 2.1 One Other Bidder

Lets say there is only one other bidder, named Bob. How much should Alice bid? She would like to bid LESS THAN 7 so she makes a profit. But there is a tradeoff:

- If she bids low then the prob of winning is low but if she wins she makes a lot.
- If she bids high then the prob of winning is high but if she wins she does not make so much.

We assume that the other player's bid is uniform— that is, prob that opponents bid is 0 is 1/9, is 1 is 1/9 etc.

So what is the right bid? We look at ALL possible bids and what there EXPECTED PROFIT is.

1. Bid is 0. The expected profit is 0.
2. Bid is 1. To find the expected profit we first need to know the prob of winning. The prob of winning is the prob that the other player bids 0, which is 1/9. If that happens then the profit is $7 - 1 = 6$. So the expected value is $\frac{1}{9} \times 6 = .666$.
3. Bid is 2. To find the expected profit we first need to know the prob of winning. The prob of winning is the prob that the other player bids 0 OR bids 1, which is 2/9. If that happens then the profit is $7 - 2 = 5$. So the expected value is $\frac{2}{9} \times 5 = 1.11$.
4. Bid is 3. The prob of winning is the prob that the other player bids 0,1,2 which is 3/9 or 1/3. If that happens then the profit is $7 - 3 = 4$. So the expected value is $\frac{1}{3} \times 4 = 1.33$.
5. Bid is 4. The prob of winning is the prob that the other player bids 0,1,2,3 which is 4/9. If that happens then the profit is $7 - 4 = 3$. So the expected value is $\frac{4}{9} \times 3 = 1.33$.
6. Bid is 5. The prob of winning is the prob that the other player bids 0,1,2,3,4 which is 5/9. If that happens then the profit is $7 - 5 = 2$. So the expected value is $\frac{5}{9} \times 2 = 1.11$.
7. Bid is 6. The prob of winning is the prob that the other player bids 0,1,2,3,4,5 which is 6/9 or 2/3. If that happens then the profit is $7-6 = 1$. So the expected value is $\frac{2}{3} \times 1 = .666$.
8. Bid is 7 or 8 or 9. Stupid- profit will be 0 or even negative.

SO the optimal bid is either 3 or 4, which give the expected payoff of 1.33. Note that it is WISE to NOT bid 7.

We could have done this faster. If the bid is $x$ then the expected profit is

$$\text{Prob of Alice winning } \times (7-x) = \text{ Prob Bob bid is in } \{0, 1, \ldots, x-1\} \times (7-x) =$$

$$\frac{x(7-x)}{9}.$$

How to maximize? Calculus! (Is there anything it can't do !)

$$f(x) = \frac{x(7-x)}{9}$$

We use product rule for derivatives. We DO NOT multiply out $x(7-x)$. Here it doesn't matter much, but in later calculations we'll see this is a good way to do it.

$$f'(x) = \frac{x(-1) + (7-x)}{9}$$

Set to zero to get $7 - 2x = 0$, so $x = 3.5$. Note that both $x = 3$ and $x = 4$ ARE the answers we got above.

Lets say Alice bids 3.5 (even though this is not an integer).

- If Alice is the higest bid then the seller gets 3.5.
- How to maximize? Calculus! (Is there anything it can't do !) What does Alice EXPECT to pay (note that she might not be the highest bidder). Her expected payout is Pr( she wins ) $\times 3.5$ which is $\frac{4}{9} \times 3.5 = 1.55$.

## 2.2 Two Other Bidders

What if there are TWO other bidders, named Bob and Carol? The key is, what is the prob that Bob and Carol BOTH bid $\le x$. As noted above, the prob that Bob bids $\le x$ is $x/9$. So the prob that BOTH do is $(x/9)^2$.

SO if Alice bids $x$ then the expected profit is
$f(x) = \frac{x^2(7-x)}{81}$

$f'(x) = \frac{-x^2+2x(7-x)}{81}$

$$\begin{aligned} -x^2 + 2x(7-x) &= 0 \\ -x + 2(7-x) &= 0 \\ 2(7-x) &= x \\ 14 - 2x &= x \\ 3x &= 14 \\ x &= 14/3 = 4.66 \end{aligned}$$

So the optimal bid is either 4 or 5.
If the bid is 4 then the expected profit is $\frac{16(7-4)}{81} = \frac{48}{81} = 0.592$.
If the bid is 5 then the expected profit is $\frac{25(7-5)}{81} = \frac{50}{81} = 0.617$.
SO the optimal bid is 5.

Note that the more people there are, the higher the bid has to be, and the lower the Expected Profit.

Lets say Alice bids 5.

- If Alice is the higest bid then the seller gets 5.
- What does Alice EXPECT to pay (note that she might not be the highest bidder). Her expected payout is Pr( she wins ) $\times$ 5 which is $(\frac{4}{9})^2 \times 5 = 0.987$.

## 2.3 $n$ Other Bidders, Generalized

The set of bids is

$$B = \{0, 1, \ldots, b-1\}.$$

Alice values the item at $v \in B$. There are $n$ other players. Let $x$ be what Alice SHOULD bid.

If $x$ is bid then the prob that Alice WINS the bid is $(\frac{x}{b+1})^n$. So the expected profit is

$$f(x) = \left(\frac{x}{b+1}\right)^n (v-x) = \frac{1}{(b+1)^n}(x^n(v-x))$$

We take the derivative

$$f'(x) = \frac{1}{(b+1)^n}(-x^n + (v-x)nx^{n-1})$$

We set this equal to 0.

$$\begin{aligned} -x^n + (v-x)nx^{n-1} = &\ 0 \\ (v-x)nx^{n-1} = &\ x^n \\ (v-x)n = &\ x \\ vn - xn = &\ x \\ vn = &\ x(1+n) \\ x = &\ \frac{vn}{n+1} \end{aligned}$$

Ignoring issues of the bid not being integral we'll say that he seller gets $\frac{vn}{n+1}$.

We plug in this value to get an approx to what the expected profit is (its not quite the answer since we are plugging in a non-integer, but its close).

$$\frac{1}{(b+1)^n}(\left(\frac{vn}{n-1}\right)^n \left(\frac{vn}{n-1} - v)\right) = \frac{v^{n+1}}{(b+1)^n}\left(\frac{n}{n-1})^n \left(\frac{1}{n-1}\right).$$

$$= \left(\frac{v}{n-1}\right)^{n+1} \left(\frac{n}{b+1}\right)^n.$$

Lets say Alice bids $\frac{vn}{n+1}$.

- If Alice is the higest bid then the seller gets $\frac{vn}{n+1}$.
- What does Alice EXPECT to pay (note that she might not be the highest bidder). Her expected payout is Pr( she wins ) $\times \frac{vn}{n+1}$ which is $(\frac{v}{b+1})^n \times \frac{vn}{n+1}$.

# 3 Second Price Auction

The WINNER of the bid pays what the SECOND largest bid was. In this case, what bids make sense?

Lets say the bids MUST be in the set

$$B = \{0, 1, 2, 3, 4, 5, 6, 7, 8\}.$$

Lets say the item is worth 7 to Alice. Lets say there is only ONE other bidder (this is the only thing that we will change later).

How much should Alice Bid? UNLIKE first price auction, bidding 7 might make sense since she will NOT pay seven. The tradeoffs presented in First Price Auctions might not quite work here since Alice may bid $x$ and pay far less than $x$. Or not.

We assume that the other player's bid is uniform— that is, prob that opponents bid is 0 is 1/9, is 1 is 1/9 etc.

So what is the right bid? We look at ALL possible bids and what there EXPECTED PROFIT is.

1. Bid is 0. The expected profit is 0.
2. Bid is 1. Unlike the first price auction, it MATTERS exactly what Bob bid, and not just that Alice wins. Prob that Alice bid 0 is 1/9, and in that case Bob gets the item and pays NOTHING. So the expected profit is

$$\frac{1}{9}(7-0) = \frac{7}{9} = 0.77$$

3. Bid is 2. Expected profit is

$$\frac{1}{9}(7-0) + \frac{1}{9}(7-1) = \frac{13}{9} = 1.44$$

4. Bid is 3.

$$\frac{1}{9}(7-0) + \frac{1}{9}(7-1) + \frac{1}{9}(7-2) = \frac{18}{9} = 2$$

5. Bid is 4.

$$\frac{1}{9}(7-0) + \frac{1}{9}(7-1) + \frac{1}{9}(7-2) + \frac{1}{9}(7-3) = \frac{22}{9} = 2.44$$

6. Bid is 5.

$$\frac{1}{9}(7-0) + \frac{1}{9}(7-1) + \frac{1}{9}(7-2) + \frac{1}{9}(7-3) + \frac{1}{9}(7-2) = \frac{22}{9} = 2.77$$

7. Bid is 6.

$$\frac{1}{9}(7-0)+\frac{1}{9}(7-1)+\frac{1}{9}(7-2)+\frac{1}{9}(7-3)+\frac{1}{9}(7-2)+\frac{1}{9}(7-1)=\frac{22}{9}=3.44$$

8. Bid is 7.

$$\frac{1}{9}(7-0)+\frac{1}{9}(7-1)+\frac{1}{9}(7-2)+\frac{1}{9}(7-3)+\frac{1}{9}(7-2)+\frac{1}{9}(7-1)+\frac{1}{9}(7)=\frac{22}{9}=4.22$$

9. Bid is 8. Still get 4.22 since if someone bids 7, Alice pays 7, and gets no profit.

Note that there is NO downside to bidding 7. More generally, Second Price auctions induce people to, for there own benefit, bid what they actually think its worth.

This is true for more people as well. More generally- continuous distributions, other discrete distributions, none of this matters. Still induced to bid what you think its worth.