Number of Cuts
Exposition by William Gasarch

1 Introduction

Throughout this exposition (1) the term *protocol* means *proportional cake cutting protocol*, and (2) we only look at the worst case for number-of-cuts.

In the COME LATE protocol the number of cuts for 3 people is 5 (in all cases). In the TRIM protocol the number of cuts for 3 people is 3 (in the worst case). The DC protocol uses 3 cuts (in all cases).

Is there a protocol for 3 people that takes 2 cuts? We show that there is not. What about other numbers-of-people and number-of-cuts? In this exposition we show the following

1. For 3 people, 3 cuts are necessary and sufficient.
2. For 3 people, 4 cuts are necessary.
3. For n people, n cuts are necessary.
4. What about a bigger cake?
5. For 4 people, 4 cuts are sufficient.

2 $n = 3$: You Need Exactly 3 Cuts

Theorem 2.1

1. *There is a 3-person protocol that only uses 3 cuts. (This is the TRIM protocol so we do not prove it here.)*
2. *Any protocol for 3 people must use at least 3 cuts.*

Proof:

Assume, by way of contradiction, that there is a protocol for 3 people using 2 cuts. We create a scenario where this does not work. KEY: we have NO control over what the cutter does, but we have COMPLETE control over everyone else's tastes.

The players are Alice, Bob, and Carol. We can assume that Alice cuts first. The pieces are P_1 and P_2. Alice values P_1 at x_1 and P_2 at x_2. All we can assume is that $x_1 + x_2 = 1$. WE set Bob and Carol as in the following table.

	P_1	P_2
Alice	x_1	x_2
Bob	1/2	1/2
Carol	1/2	1/2

There are two cases. Either Alice makes the next cut or Bob does (Carol doing it is the same as Bob doing it).

Case 1: Alice takes the next cut. We can assume she cuts P_2 into two pieces, P_{21} and P_{22}. WE set Alice and Bob's valuation as in the following table:

	P_1	P_{21}	P_{22}
Alice	x_1	x_{21}	x_{22}
Bob	1/2	1/4	1/4
Carol	1/2	1/4	1/4

There are no more cuts to be made. Note that P_1 is the *only* piece acceptable to both Bob and Carol. They can't both have it! Hence the protocol fails.

Case 2: Bob takes the next cut. We can assume he cuts P_2 into two pieces, P_{21} and P_{22}. Bob values P_{21} at y_1 and P_{22} at y_2.

- We know $y_1 + y_2 = 1/2$.
- We assume $y_1 \leq y_2$, hence $y_1 \leq 1/4$ so P_{21} not acceptable to Bob.

WE set Alice and Carol's valuations as in the table below.

	P_1	P_{21}	P_{22}
Alice	x_1	0	x_2
Bob	1/2	$y_1 \leq$	y_2
Carol	1/2	1/4	1/4

There are no more cuts to be made. Note that P_{21} is not acceptable to Alice, Bob, or Carol. But someone has to take it. Hence the protocol fails.

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3 $n = 3$ With a Wee Bit More Cake

We restate Theorem 2.1

Theorem 3.1 *Any protocol for 3 people that starts off with a cake of size 1, and guarantees that everyone gets $\geq \frac{1}{3}$, uses at least 3 cuts.*

What if we started out with JUST a bit more cake?

Theorem 3.2 *There exists an $\epsilon > 0$ such that the following is true: Any protocol for 3 people that starts off with a cake of size $1 + \epsilon$, and guarantees that everyone gets $\geq \frac{1}{3}$, uses at least 3 cuts. We can take any ϵ such that $0 < \epsilon <= 1/3$. (Note that all 3 people value the entire cake at $1 + \epsilon$.)*

Proof:

We will set ϵ later.

Assume, by way of contradiction, that there is a protocol for 3 people using 2 cuts that splits a cake of size $1 + \epsilon$ into three pieces so that each person gets a piece of size $\geq 1/3$. We create a scenario where this does not work. KEY: we have NO control over what the cutter does, but we have COMPLETE control over everyone else's tastes.

The players are Alice, Bob, and Carol. We can assume that Alice cuts first. The pieces are P_1 and P_2. Alice values P_1 at x_1 and P_2 at x_2. All we can assume is that $x_1 + x_2 = 1 + \epsilon$. WE set Bob and Carol as in the following table.

	P_1	P_2
Alice	x_1	x_2
Bob	$(1+\epsilon)/2$	$(1+\epsilon)/2$
Carol	$(1+\epsilon)/2$	$(1+\epsilon)/2$

There are two cases. Either Alice makes the next cut or Bob does (Carol doing it is the same as Bob doing it).

Case 1: Alice takes the next cut. We can assume she cuts P_2 into two pieces, P_{21} and P_{22}. WE set Bob and Carol's valuations as in the following table:

	P_1	P_{21}	P_{22}
Alice	x_1	x_{21}	x_{22}
Bob	$(1+\epsilon)/2$	$(1+\epsilon)/4$	$(1+\epsilon)/4$
Carol	$(1+\epsilon)/2$	$(1+\epsilon)/4$	$(1+\epsilon)/4$

There are no more cuts to be made.

We WANT to make P_{21} and P_{22} NOT acceptable to Bob or Carol. Hence we need

$$\begin{aligned} (1+\epsilon)/4 &< 1/3 \\ 1+\epsilon &< 4/3 \\ \epsilon &< 1/3 \end{aligned}$$

NOW note that P_1 is the *only* piece acceptable to both Bob and Carol. They can't both have it! Hence the protocol fails.

Case 2: Bob takes the next cut. We can assume he cuts P_2 into two pieces, P_{21} and P_{22}. Bob values P_{21} at y_1 and P_{22} at y_2.

- We know $y_1 + y_2 = (1+\epsilon)/2$.
- We assume $y_1 \le y_2$ hence $y_1 \le (1+\epsilon)/4 < 1/3$ so P_{22} is not acceptable to Bob.

WE set Alice and Carol's values by the following table.

	P_1	P_{21}	P_{22}
Alice	x_1	0	x_2
Bob	$(1_{+}\epsilon)/2$	$y_1 \le$	y_2
Carol	$(1+\epsilon)/2$	$(1+\epsilon)/4$	$1/4$

There are no more cuts to be made. Note that P_{21} is not acceptable to Alice, Bob, or Carol. But someone has to take it. Hence the protocol fails.

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4 What if a Stranger was allowed to cut?

We have always considered a protocol between Alice, Bob, Carol, etc to be an algorithm where every cut is made by one of the players. But what if we allow a stranger to make a cut? The proofs of Theorem 2.1 and 3.2 would need some extra cases; however, they would be easy cases since the Strangers cuts allow us to set everyones valuations (unlike, say, when Alice cuts we can't control Alice).

Henceforth we assume that protocols include strangers being allowed to cut, but will not ever bother with those cases.

5 A Different Viewpoint

We introduce a useful notation and restate a scaled version Theorem 3.2 in that notation.

Def 5.1 Let s, p be positive rationals. Let $n \in \mathsf{N}$. An (n, s, p) *protocol* is a protocol for n people which takes a cake of size s and gives everyone $\geq p$.

Note 5.2 An $(n, 1, 1/n)$ protocol is an n person proportional protocol.

Theorem 5.3 *For all y, for all $\epsilon > 0$, there is no 2-cut $(3, 4y-\epsilon, y)$ protocol.*

Proof: Assume, by way of contradiction, that there is a 2-cut $(3, 4y-\epsilon, y)$ protocol. We create a scenario where this does not work. KEY: we have NO control over what the cutter does, but we have COMPLETE control over everyone else's tastes.

The players are Alice, Bob, and Carol. We can assume that Alice cuts first. The pieces are P_1 and P_2. Alice values P_1 at x_1 and P_2 at x_2. All we can assume is that $x_1 + x_2 = 4y + \epsilon$. WE set Bob and Carol as in the following table.

	P_1	P_2
Alice	x_1	x_2
Bob	$2y - (\epsilon/2)$	$2y - (\epsilon/2)$
Carol	$2y - (\epsilon/2)$	$2y - (\epsilon/2)$

There are two cases. Either Alice makes the next cut or Bob does (Carol doing it is the same as Bob doing it).

Case 1: Alice takes the next cut. We can assume she cuts P_2 into two pieces, P_{21} and P_{22}. WE set Bob and Carol's valuations as in the following table:

	P_1	P_{21}	P_{22}
Alice	x_1	x_{21}	x_{22}
Bob	$2y - (\epsilon/2)$	$y - (\epsilon/4)$	$y - (\epsilon/4)$
Carol	$2y - (\epsilon/2)$	$y - (\epsilon/4)$	$y - (\epsilon/4)$

There are no more cuts to be made.

Since neither P_{21} nor P_{22} are acceptable to Bob or Carol they both get P_1. They can't both have it! So the protocol fails.

Case 2: Bob takes the next cut. We can assume he cuts P_2 into two pieces, P_{21} and P_{22}. Bob values P_{21} at y_1 and P_{22} at y_2.

- We know $y_1 + y_2 = 2y - (\epsilon/2)$.
- We assume $y_1 \le y_2$ hence $y_1 \le y - (\epsilon/4)$ so P_{22} is not acceptable to Bob.

WE set Alice and Carol's values by the following table.

	P_1	P_{21}	P_{22}
Alice	x_1	0	x_2
Bob	$2y - (\epsilon/2)$	$y_1 \le$	y_2
Carol	$2y - (\epsilon/2)$	$y - (\epsilon/4)$	$y - (\epsilon/4)$

There are no more cuts to be made. Note that P_{21} is not acceptable to Alice, Bob, or Carol. Hence the protocol fails.
∎

Corollary 5.4 *There is no 3-person 2-cut proportional protocol.*

Proof: If we plug $y = 1/3$ and $\epsilon = 1/3$ into Theorem 5.3 we obtain that there is no 2-cut $(3, 1, 1/3)$ protocol. ∎

6 $n = 4$: You Need at Least Four Cuts

We want to show that there is no 4-person 3-cut prop. protocol. We show something stronger which we will then use in our 5-person lower bound.

Theorem 6.1 *For all y, for all $\epsilon > 0$, there is no 3-cut $(4, 5y - \epsilon, y)$ protocol.*

Proof:
Assume, by way of contradiction, that there is a 3-cut $(4, 5y - \epsilon, y)$ protocol. KEY: we have NO control over what the cutter does, but we have COMPLETE control over everyone else's tastes.

Note that after 3 cuts there will be exactly 4 pieces. Hence if there is a piece that only one player likes, that player must get it.

The players are Alice, Bob, Carol, and Donna. We can assume that Alice cuts first. The pieces are P_1 and P_2. Alice values P_1 at x_1 and P_2 at x_2. All we can assume is that $x_1 + x_2 = 5y - \epsilon$. WE set Bob and Carol as in the table below (we determine δ later).

	P_1	P_2
Alice	x_1	x_2
Bob	$4y - (\epsilon/2)$	$y - (\epsilon/2)$
Carol	$4y - (\epsilon/2)$	$y - (\epsilon/2)$
Donna	$4y - (\epsilon/2)$	$y - (\epsilon/2)$

Alice is the only one who likes P_2. Hence Alice will get P_2. Consider the rest of the protocol. It is a 3-person protocol with Bob, Carol, Donna (if Alice cuts she can be regarded as a stranger).

They are splitting a cake of size $4y - (\epsilon/2)$.

They are each getting y.

They are only using 2 cuts.

Hence they have a 2-cut $(3, 4y - (\epsilon/2), y)$ protocol.

This contradicts Theorem 5.3.

∎

Corollary 6.2 *There is no 4 person 3-cut prop. protocol.*

Proof: If we plug $y = 1/4$ and $\epsilon = 1/4$ into Theorem 6.1 we obtain that there is no 3-cut $(4, 1, 1/4)$ protocol. ∎

We leave it to the reader to show, from Theorem 6.1 that for all y, for all $\epsilon > 0$, there is no 4-cut $(5, 6y - \epsilon, y)$ protocol.

7 General n

Theorem 7.1 *For all n, for all y, for all $\epsilon > 0$, there is no $(n-1)$-cut $(n, (n+1)y - \epsilon, y)$ protocol.*

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Corollary 7.2 *There is no n person $(n-1)$-cut prop. protocol.*

Proof: If we plug $y = 1/n$ and $\epsilon = 1/n$ into Theorem 7.1 we obtain that there is no $(n-1)$-cut $(n, 1, 1/n)$ protocol. ∎

8 How Big a Cake Do we Need?

9 $n = 4$: We CAN do 4 people in 4 cuts

NOTE- I DID NOT COVER THIS IN CLASS AND WILL NOT. IT IS HERE JUST SO YOU CAN SEE HOW COMPLICATED THESE PROTCOLS CAN GET.

Note the following

- The COME LATE protocol for 4 people uses 14 cuts.
- The TRIM protocol for 4 people uses 7 cuts.
- The DC protocol for 4 people uses 5 cuts.

Hence none of these achieve 4 cuts. The good news is that there is a 4 person 4-cut protocol. The bad news is that its a bit detailed and does not generalize. The good news is that if you use it for the $n = 4$ case of any of the above protocols they will improve— slightly.

Theorem 9.1 *There is a 4-person, 4-cut protocol.*

Proof:

The players will be Alice, Bob, Carol, and Donna. The pieces of cake will be P with some subscripts. We use the phrase *Alice thinks* $P_1 < P_2$ to mean that she values P_2 more than P_1.

There are two key intuitions:

- If the cake is in four pieces P_1, P_2, P_3, P_4 and Alice and Bob both think that $P_1 \le P_2$ and $P_3 \le P_4$ then they think that $P_2 \cup P_4$ has value $\ge 1/2$.
- If the cake is in four pieces P_1, P_2, P_3, P_4 and Alice and Bob both think that $P_1 \le P_2$ and $P_3 \le 1/4$ then they think that $P_2 \cup P_4$ has high value since they are getting rid of two pieces that are small (this is informal).

We present the protocol. There are many cases and we will present the easier cases first.

1. Alice cuts the cake (evenly). Bob, Carol, and Donna all write down which piece they think is bigger (honestly), with ties broken arbitrarily (honestly). We denote the pieces P_1 and P_2. One cut.

2. **Case 1:**

   - Bob and Carol think $P_1 \geq 1/2$
   - Donna thinks $P_2 > 1/2$.

   (Any case where two of them prefer one of the pieces, and the other person prefers the other piece, is similar and omitted.) Then we do the following:

   (a) Bob and Carol do Cut-and-Choose with P_1

   (b) Alice and Donna do Cut-and-Choose with P_2.

   WE ARE DONE. Three Cuts TOTAL.

3. **Case 2:**

   - Bob and Carol and Donna all think $P_1 > 1/2$. (The case where they all think $P_2 > 1/2$ is similar.) Alice cuts P_2 (evenly). The following table summarizes what we know.

	P_1	P_{21}	P_{22}
Alice	$1/2$	$1/4$	$1/4$
Bob	$> 1/2$	x_1	x_2
Carol	$> 1/2$	y_1	y_2
Carol	$> 1/2$	z_1	z_2

   - $x_1 + x_2 < 1/2$
   - $y_1 + y_2 < 1/2$
   - $z_1 + z_2 < 1/2$.

   Even so, its possible that we get lucky and Bob, Carol, or Donna thinks that one of these pieces is $\geq 1/4$— this is Case 2.1 below.

   (a) **Case 2.1**

   - Bob thinks $x_1 \geq 1/4$ (The cases where Bob thinks $x_2 \geq 1/4$, Carol thinks $y_1 \geq 1/4$, Carol thinks $y_2 \geq 1/4$, Donna thinks $z_1 \geq 1/4$, Donna thinks $z_2 \geq 1/4$, are similar.) The following table summarizes what we know.

	P_1	P_{21}	P_{22}
Alice	$1/2$	$1/4$	$1/4$
Bob	$> 1/2$	$x_1 > 1/4$	x_2
Carol	$> 1/2$	y_1	y_2
Donna	$> 1/2$	z_1	z_2

i. Alice takes P_{22}

ii. Bob takes P_{21}

iii. Carol and Donna do cut and choose on P_1 which they both think is $> 1/2$.

WE ARE DONE. THREE CUTS.

(b) **Case 2.2** $x_1, x_2, y_1, y_2, z_1, z_2 < 1/4$. Consider the questions:

- $x_1 \le x_2$?
- $y_1 \le y_2$?
- $z_1 \le z_2$?

Two of these must give the same answer. We assume

- $x_1 \ge x_2$
- $y_1 \ge y_2$
- We make NO assumption about z_1 and z_2.

The other cases where two of them give the same answer are similar.

i. Alice takes P_{22} and goes away.

ii. We will still keep P_{22} in the table for we will need the fact that Bob and Carol think its small. Bob cuts P_1 (evenly).

	P_{11}	P_{12}	P_{21}	P_{22}
Bob	$> 1/4$	$> 1/4$	$x_1 \ge$	x_2
Carol	w_1	w_2	$y_1 \ge$	y_2
Donna	u_1	u_2	$z_1 < 1/4$	$z_2 < 1/4$

We know the following

- $w_1 + w_2 > 1/2$
- $u_1 + u_2 > 1/2$
- $x_1 \ge x_2$.
- $y_1 \ge y_2$.
- $x_1, x_2, y_1, y_2, z_1, z_2 < 1/4$.

We have several subcases. In the first one Carol and Donna both have a different piece that they like so its easy. The others are based on if Carol and Donna agree about which of P_{11} and P_{12} is bigger.

At this point 3 cuts have been made. In all cases below an additional one cut is made so there will be 4 total.

i. **Case 2.2.1** Carol thinks $P_{11} > P_{12}$ and Donna thinks $P_{12} > P_{11}$.
KEY: Carol and Bob both thinks $P_{11} \geq P_{12}$ and $P_{21} \geq P_{22}$. Hence they think $P_{11} \cup P_{21} \geq 1/2$. They split it using cut and choose.
KEY: Donna thinks $P_{12} > P_{11}$ and that $P_{11} + P_{12} > 1/2$. Hence she thinks $P_{12} > 1/4$ and takes it.
WE ARE DONE. FOUR CUTS.

ii. **Case 2.2.2** Carol and Donna both think $P_{11} > P_{12}$. We show that Carol and Donna both think $P_{11} + P_{21} \geq 1/2$, hence they can split it. But first we look at Bob.
Bob thinks $P_{12} \geq 1/4$. He takes it.
KEY: Carol thinks $P_{11} > P_{12}$ and $P_{21} \geq P_{22}$. Hence she thinks $P_{11} + P_{21} \geq 1/2$.
KEY: Intuition of what Donna thinks: She thinks $P_{11} > P_{12}$ and that $P_{22} < 1/4$. Hence she should think that $P_{11} \cup P_{21}$ is big since it is the result of discarding two *small* pieces. More formally:

$$P_{11} + P_{12} + P_{21} + P_{22} = 1$$

Donna thinks $P_{22} < 1/4$, hence Donna thinks:

$$P_{11} + P_{12} + P_{21} = 1 - P_{22} > 3/4.$$

Donna thinks

$$P_{11} + P_{12} = \alpha > 1/2$$

and

$$P_{11} > P_{12}.$$

Hence Donna thinks

$$P_{12} < \alpha/2.$$

Hence

$$P_{11} + P_{21} = 1 - P_{22} - P_{12} > 1 - \frac{1}{4} - \frac{\alpha}{2} = \frac{3}{4} - \frac{\alpha}{2} = \frac{3 - 2\alpha}{4}.$$

Since $\alpha > 1/2$ we have that Donna thinks

$$P_{11} + P_{21} > 1/2$$

Have Carol and Donna do Cut and choose on $P_{11} + P_{21}$. WE ARE DONE. FOUR CUTS.

∎

So we can do $n = 4$ with 4 cuts. Can we do better? NO

10 What Else is Known

For $n = 5$ there is a 6 cut protocol but no 5 cut protocol. The proofs are very messy.