HW 4 HONR 209M. Morally DUE Tuesday Oct 1

SOLUTIONS

1. (0 points) What is your name? Write it clearly. Staple your HW. When is the first midterm? When is the final?

   NOTE- THIS HW IS TWO PAGES, DO NOT MISS SECOND PAGE.

2. (60 points) Alice and Bob are looking at cake that we think of as the interval $[0,1]$. Let $f(x) = 2x$ and $g(x) = 1$. Alice's valuation is $v(a,b) = \int_a^b f(x) = b^2 - a^2$. Bob's valuation is $v(a,b) = \int_a^b g(x) = b - a$.

   (a) Assume that neither knows the others valuation. Assume they do Cut and Choose with Alice cutting, Bob Choosing. Where is the cut going to be? How much does Alice get? How much does Bob get? What is Alice + Bob (the total)?

   (b) Assume Alice knows Bob's valuation. Assume they do Cut and Choose with Alice cutting, Bob Choosing. Where should Alice cut to do BETTER than if she didn't know (which is the case in part a)? How much does Alice get? How much does Bob get? What is Alice + Bob (the total)? (NOTE- there are many answers. Pick one where Alice does BETTER than she does in part (a). Try to make it much better but don't try to optimize it as this is actually not possible.)

   (c) If Alice and Bob reveal their honest valuation to each other and agree to choose a value x to cut at so that they both have the same amount, What x do they choose? How much does Alice get? How much does Bob get? How much does Alice get? How much does Bob get? What is Alice + Bob (the total)?

   (d) If they both reveal their honest valuation to each other and agree to choose a value x to cut at so that the SUM of what Alice and Bob gets is maximized, what x do they choose, How much does Alice get? How much does Bob get? What is the TOTAL of what Alice gets PLUS what Bob gets? (If its less than the TOTAL in part c then your answer is wrong.)

SOLUTION TO PROBLEM 2

a) Alice cuts at a such that $a^2 = 1/2$, so she cuts at $\sqrt{2}/2 \equiv 0.707$ Bob takes the LEFT part.

Alice gets $\frac{1}{2} = 0.5$ (the cutter ALWAYS gets 1/2)

Bob gets approx 0.707.

TOTAL: approx 1.207.

b) If Alice knows Bob's valuation she will cut very close to Bob's 1/2 point, but shaded just a bit to the right so that Bob will take the left piece, and Alice gets the right piece. (Alice wants the right piece since its more valuate to her.) Alice should cut at $\frac{1}{2} + \epsilon$ where we pick ϵ later but it will be small.

Bob gets $[1, \frac{1}{2} + \epsilon]$. NOTE- just a wee bit more than $\frac{1}{2}$.
Alice gets $[\frac{1}{2}+\epsilon, 1]$ so gets $1^2 - (\frac{1}{2}+\epsilon)^2 = 1 - (1/4 + \epsilon + \epsilon^2) = 3/4 - \epsilon - \epsilon_2)$. NOTE- just a wee bit less than 3/4.

c) If the cut is at $x \in [0, 1]$ then, giving Alice the RIGHT piece and Bob the LEFT piece:

Alice gets $[x, 1]$ so gets $1 - x^2$

Bob gets $[0, x]$ so gets x.

$$\begin{aligned} 1 - x^2 = &\ x \\ -x^2 - x + 1 = &\ 0 \\ x^2 + x - 1 = &\ 0 \\ x = &\ \tfrac{-1+\sqrt{5}}{2} \end{aligned}$$

Alice gets $1 - x^2 \sim 0.618$

Bob gets the same.

TOTAL = 1.236.

d) We want to maximize

$$(1 - x^2) + x = -x^2 + x + 1 \text{ on the interval } [0, 1]$$

The derivative is $-2x + 1$. So an extreme happens when $-2x + 1 = 0$ or when $x = 1/2$. The second derivative is -2, so the extreme is a max.

Alice gets $[1/2, 1]$, so she gets $1 - (1/2)^2 = 3/4$.

Bob gets $[0, 1/2]$, so he gets $1/2$.

TOTAL is $3/4 + 1/2 = 1.25$.

3. (40 points) Alice and Bob want to split a cake in the ratio $(121 : 100)$. (For this problem you can assume that for $(a : 1)$ the number of cuts is $\lceil \lg(a+1) \rceil$ and for $(1 : b)$ its $\lceil \lg(b+1) \rceil$. Hence you can stop the tree if either number is 1.)

   (a) Give the protocol, in the form of a tree, for when Alice and Bob do the near-halves protocol (reducing when it is possible). How many cuts does it take in the worst case?

   (b) Give a protocol where the FIRST step is that Bob cuts the cake in ratio $(1 : 220)$ and either Alice or Bob takes the 1-piece, and after that do the near-halves protocol. How many cuts does it take in the worst case?

SOLUTION to Problem 3.

Can't draw trees that well, so you should do that yourself but I will present the ideas, and do the first step of both.

a) If you do the near-halves on (121:100) then the first cut will be determined by near-halfing $121 + 100 = 221$. This is (110:111). RECALL- in the near halves alg the player who is due less cuts it in near half and OFFERS the other player EITHER piece. He takes the one that is as-advertised.

If he takes the 110 then we are at (121-110:100)=(21:100).

If he takes the 111 then we are at (121-111:100)=(20:100)=(1:5). From (1:5) we get $\lceil \lg(6) \rceil = 3$. So THAT side does 4 cuts. The other side we would need to repeat this process on.

b) We are asked to make have one player do a VERY SMALL cut and offer it to the other who will either take it or give it to the cutter. VERY SMALL in this case (1:220). Why SO SMALL a cut? We'll see that it leads to big reductions no BOTH sides.

If Alice takes it then we are at (121-1:100)=(120:100)=(6:5)

If Bob teaks it then we are at (121:100-1)=(121:99)=(11:9)

We then to near-halves on both sides, but we will get FEWER cuts than in part a.