HW 5 HONR 209M. Morally DUE Tuesday Oct 8

SOLUTIONS

1. (0 points) What is your name? Write it clearly. Staple your HW. When is the first midterm? When is the final?

   NOTE- THIS HW IS TWO PAGES, DO NOT MISS SECOND PAGE.

2. (60 points) Alice and Bob are looking at cake that we think of as the interval $[0, 1]$. Let $f(x) = 2x$ and $g(x) = x + \frac{1}{2}$. Alice's valuation is $v_A(a, b) = \int_a^b f(x)dx = b^2 - a^2$. Bob's valuation is $v_B(a, b) = \int_a^b g(x)dx = \frac{b^2-a^2}{2} + \frac{b-a}{2}$

   (a) Find the number x_{LA} such that if the cut is at x_{LA} and Alice takes the interval $[0, x_{LA}]$ then Alice gets exactly $\frac{1}{3}$. Note that if $x \geq x_{LA}$ and Alice takes $[0, x]$ then she has $\geq \frac{1}{3}$. (We call it x_{LA} since Left piece is going to Alice.)

   (b) Find the number x_{RA} such that if the cut is at x_{RA} and Alice takes the interval $[x_{RA}, 1]$ then Alice gets exactly $\frac{1}{3}$. Note that if $x \leq x_{RA}$ and Alice takes $[x, 1]$ then she has $\geq \frac{1}{3}$. (We call it x_{RA} since Right piece is going to Alice.)

   (c) Find the number x_{LB} such that if the cut is at x_{LB} and Bob takes the interval $[0, x_{LB}]$ then Bob gets exactly $\frac{2}{3}$. Note that if $x \geq x_{LB}$ and Bob takes $[0, x]$ then he has $\geq \frac{2}{3}$. (We call it x_{LB} since Left piece is going to Bob.)

   (d) Find the number x_{RB} such that if the cut is at x_{RB} and Bob takes the interval $[x_{RB}, 1]$ then Bob gets exactly $\frac{2}{3}$. Note that if $x \leq x_{RB}$ and Bob takes $[x, 1]$ then he has $\geq \frac{2}{3}$. (We call it x_{RB} since Right piece is going to Bob.)

   (e) Find the set of ALL x such that if the cut is at x and Alice takes the left and Bob takes the right, Alice gets $\geq \frac{1}{3}$ and Bob get $\geq \frac{2}{3}$.

   (f) Find the set of ALL x such that if the cut is at x and Alice takes the right and Bob takes the left, Alice gets $\geq \frac{1}{3}$ and Bob get $\geq \frac{2}{3}$.

SOLUTION TO PROBLEM 2

a) If the cuts is at x and Alice gets $[0, x]$ then she gets x^2. Hence we need to know when $x^2 = 1/3$:

$$x = \sqrt{\frac{1}{3}} = \frac{\sqrt{3}}{3} \sim 0.577.$$

And we note for later that $(\forall x \geq 0.577)[v_A(0, x) \geq 1/3]$.

b) If the cuts is at x and Alice gets $[x, 1]$ then she gets $1 - x^2$. Hence we need to know when $1 - x^2 = 1/3$:

$$1 - x^2 = 1/3$$

$$x^2 = 2/3$$

$$x = \sqrt{\frac{2}{3}} = 0.816$$

And we note for later that $(\forall x \leq 0.816)[v_A(x, 1) \geq 1/3]$.

c) If the cuts is at x and Bob gets $[0, x]$ then he gets $(x^2 + x)/2$. Hence we need to know when $(x^2 + x)/2 = 2/3$:

$$x^2 + x - \frac{4}{3} = 0$$

$$x = \frac{-1 \pm \sqrt{1 + 4 \times 1 \times \frac{4}{3}}}{2} = \frac{-1 \pm \sqrt{19/3}}{2} = 0.758$$

And we note for later that $(\forall x \geq 0.758)[v_B(0, x) \geq 2/3]$.

d) If the cuts is at x and Bob gets $[x, 1]$ then he gets $\frac{1-x^2}{2} + \frac{1-x}{2}$.

$$\frac{1 - x^2}{2} + \frac{1 - x}{2} = 2/3$$

$$1 - x^2 + 1 - x = 4/3$$

$$2 - x^2 - x = 4/3$$

$$x^2 + x - 2/3$$

$$x = \frac{-1 \pm \sqrt{1 + 4 \times 1 \times 2/3}}{2} = \frac{-1 \pm \sqrt{11/3}}{2}$$

3. (40 points) Alice, Bob, Carol, and Donna want to split cake in ratio $(a : b : c : d)$. Give a protocol for this. (HINT: The first step is to use the Alice-Bob-Carol ratio $(a : b : c)$.

   SOLUTION TO PROBLEM 3.

   (a) Alice, Bob, Carol cut the cake in ratio $(a : b : c)$. Note that Alice has $\frac{a}{a+b+c}$, Bob has $\frac{b}{a+b+c}$ and Carol has $\frac{c}{a+b+c}$.
   (b) Alice and Donna split Alice's piece in ratio $(a + b + c : d)$ Note that Alice has $\frac{a}{a+b+c} \times \frac{a+b+c}{a+b+c+d} = \frac{a}{a+b+c+d}$ and Donna gets $\frac{a}{a+b+c} \times \frac{d}{a+b+c+d}$.
   (c) Bob and Donna split Bob's piece in ratio $(a+b+c : d)$ Note that Bob has $\frac{b}{a+b+c} \times \frac{a+b+c}{a+b+c+d} = \frac{b}{a+b+c+d}$ and Donna gets $\frac{b}{a+b+c} \times \frac{d}{a+b+c+d}$.
   (d) Carol and Donna split Carol's piece in ratio $(a + b + c : d)$ Note that Carol has $\frac{c}{a+b+c} \times \frac{a+b+c}{a+b+c+d} = \frac{c}{a+b+c+d}$ and Donna gets $\frac{c}{a+b+c} \times \frac{d}{a+b+c+d}$.

   From the comments made in the protocol we can see that Alice, Bob, and Carol are getting their unfair-share.

   Donna gets:

$$\frac{a}{a+b+c} \times \frac{d}{a+b+c+d} + \frac{b}{a+b+c} \times \frac{d}{a+b+c+d} + \frac{c}{a+b+c} \times \frac{d}{a+b+c+d} =$$

$$\left(\frac{a}{a+b+c} + \frac{b}{a+b+c} + \frac{c}{a+b+c}\right)\frac{d}{a+b+c+d} = 1 \times \frac{d}{a+b+c+d}$$