Linear Valuations
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1 Introduction

We always view a cake as being the interval $[0,1]$.

In class we have considered valuations as follows.

(SEE PICTURE IN CLASS)

Let $f(x)$ be defined as follows:

$$f(x) = \begin{cases} 4/3 \text{ if } 0 \le x \le 3/4 \\ 0 \text{ otherwise} \end{cases} \tag{1}$$

Then

$$val(a,b) = \int_a^b f(x)dx.$$

This is the VALUE that Alice gives to the slice of cake defined by a and b as the endpoints.

(Recall that $\in_a^b f(x)dx$ is the area under the curve $f(x)$ in between a and b.)

Note that $val(0,1) = 1$.

What properties of $f(x)$ make this measure make sense? All we need is that $f(x)$ is continuous where it is defined and that $\int_0^1 f(x)dx = 1$

For most of this document we will look at $f(x)$ linear.

2 One Large Example

We will consider the following two valuations defined by the function f and g on $[0,1]$.

Alice has

$$f(x) = \begin{cases} 9x/2 \text{ if } 0 \le x \le 2/3 \\ 0 \text{ otherwise} \end{cases} \tag{2}$$

$$g(x) = \begin{cases} -32x/9 + 32/9 \text{ if } 1/4 \le x \le 1 \\ 0 \text{ otherwise} \end{cases} \tag{3}$$

SEE PICTURE IN CLASS.

One can check that $\int_0^1 f(x)dx = \int_0^1 g(x)dx = 1$. You do not need calculus- its just the area of a triangle.

We go through many ways to divide the cake.

1) CUT AND CHOOSE- Alice cuts and Bob Chooses.

We need an a such that $\int_0^a f(x)dx = 1/2$. Don't really need calculus since its just triangles. Note that

$\int_0^a f(x)dx = 9a^2/2$.

Hence take $a = \frac{\sqrt{2}}{3} \sim 0.471$ Note that $0.471 > 1/4$ which makes Bob's choice easier to calculate. Bob either chooses the right half: $\int_a^1 -32x/9 + 32/9$ or the left half: $1 - \int_a^1 -32x/9 + 32/9$

We do this via the area-of-a-triangle and not calculus.

$$\frac{1}{2}(1-a)(-32a/9 + 32/9) = (1-a)(-16a/9 + 16/9) = \frac{16}{9}(1-a)^2$$

With $a = \frac{\sqrt{2}}{3}$ the left side has value to Bob of approx 0.4974 The right side has value $1 - 0.4974 = .5025$. Hence Bob takes the right side.

- ALICE gets 1/2 (the cutter always does)
- BOB gets 0.5025.
- TOTAL: 1.5025.

2) Both reveal their evaluation function and find a cut point a that gives them both the same.

Lets say a is the cut point. EITHER Alice gets LEFT and Bob gets RIGHT or vice versa. Looking at the picture clearly you want to give Bob the LEFT and Alice the RIGHT.

If cut point is a and Alice gets RIGHT, Bob gets left, then

- Alice gets $1 - 9a^2/4$.
- Bob gets $1 - 16(1-a)^2/9$

EQUATE these two to get

$$\begin{aligned} 9a^2/4 &= 16(1-a)^2/9 \\ 81a^2 &= 64(a-1)^2 \\ 9a &= 8(1-a) \\ 9a &= 8 - 8a \\ 17a &= 8 \\ a &= 8/17 \end{aligned}$$

- Alice gets $1 - \frac{9}{4} \times \frac{64}{289} = 1 - \frac{9\times16}{289} \sim 0.5017$.
- Bob gets the same.
- Total: 1.0034.

This seems awfully small! Why?

Because- WE GAVE ALICE SOME STUFF SHE REGARDS AS ZERO and WE GAVE BOB SOME STUFF HE REGARDS AS ZERO.

How to do this right? See next Method

3) EQUALIZE done right.

First give Alice $[0, 1/4]$ so she has $\frac{1}{2} \times \frac{1}{4} \times \frac{9}{2} \times \frac{1}{4} = \frac{9}{64}$.

Second give Bob $[2/3, 1]$ so he has $\frac{1}{2} \times \frac{1}{3}\frac{32}{9}(1 - 2/3) = \frac{1}{3}\frac{16}{9} \times \frac{1}{3} = \frac{16}{81}$.

We now want to find an $a \in [1/4, 2/3]$ so that if we cut the cake at a and give the LEFT to Alice and the RIGHT to Bob then they NOW have it equal. OR we could give the RIGHT to Alice and the LEFT to Bob. We do whichever gives them both more. We set up the equation but do not solve it, for Alice-LEFT, Bob-RiGHT.

If cut at a and give Alice the Left part then she gets (in addition to her $\frac{9}{64}$) the area of the trapezoid bounded by the x-axis, the lines $x = 1/4$ and $x = a$, and the line $y = \frac{9x}{2}$. This area is the area of the obv rectangle plus a small triangle:

$$A(a) = (a - \frac{1}{4}) \times \frac{9a}{2} + \frac{1}{2} \times (a - \frac{1}{4}) \times \frac{9}{4}.$$

We do a similar calculation for Bob to obtain $B(a)$. We then set

$$\frac{9}{64} + A(a) = \frac{16}{81} + B(a).$$

and obtain a value for a that will equalize both sides.

4) Maximize TOTAL

First give Alice $[0, 1/4]$ so she has $\frac{1}{2} \times \frac{1}{4} \times \frac{9}{2} \times \frac{1}{4} = \frac{9}{64}$.

Second give Bob $[2/3, 1]$ so he has $\frac{1}{2} \times \frac{1}{3}\frac{32}{9}(1 - 2/3) = \frac{1}{3}\frac{16}{9} \times \frac{1}{3} = \frac{16}{81}$.

We then pick a value of a to maximize the total. In particular, using the analysis in algorithm 3, we find $a \in [1/4, 2/3]$ to maximize $A(a) + B(a)$

3 Generalize

Everything we did in the last section could be done with general functions f, g so long as they integrate to 1 and are easy enough to find areas under curves. The algebra may get rough but if we are happy with approximations that should not be a problem.

We formalize some of the algorithms in the last section.

1) CUT AND CHOOSE-SAME AS USUAL.

2) EQUALIZE!

1. Simultaneously Alice reveals f and Bob reveals g.
2. Give Alice stuff that Bob finds worthless.
3. Give Bob stuff that Alice finds worthless.
4. Let $[c, d]$ be the interval that is left.
5. They find the value a such that $\int_c^a f(x)dx = \int_a^d g(x)dx$ They find the value b such that $\int_c^b g(x)dx = \int_b^d f(x)dx$ Use whichever one maximizes both of their values.

3) MAX TOTAL!

1. Simultaneously Alice reveals f and Bob reveals g.
2. Give Alice stuff that Bob finds worthless.
3. Give Bob stuff that Alice finds worthless.
4. Let $[c, d]$ be the interval that is left.
5. They find the value a such that $\int_c^a f(x)dx + \int_a^d g(x)dx$ is maximized. They find the value b such that $\int_c^b f(x)dx + \int_b^d g(x)dx$ is maximized. Use whichever of a, b has a higher total value.
6. If one of the players gets $< 1/2$ then he or she volunteers to cut the cake. They then use the knife to stab the other player and take the entire cake.