Unfair Division
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1 Introduction

Whenever we say something like *Alice has a piece worth* α we mean worth α TO HER.

We have been talking about splitting cakes between n people in the ratio of $(1 : 1 : \cdots : 1)$. What if we want an UNFAIR division? Say we want Alice to get 3/5 of the cake (or more) and Bob to get 2/5 (or more). Can we do this? How many cuts would it take? What about if we want Alice, Bob, Carol do split it $(2 : 3 : 89)$? Can we do that? The answers are YES to all of these. We will be interested in number-of-cuts.

NOTE: In class I will do those nice tree diagrams. I can't do those in text, so this text (more than most) is a SUPPLEMENT to your class notes.

2 A Naive 2-Player Protocol for Unfair Division

Theorem 2.1 *For all* a, b *there is a 2-player protocol for* $(a : b)$ *division that takes* $a + b - 1$ *cuts.*

Proof:

We assume $a \leq b$, though if $a = b$ we just do cut and choose.

1. Alice cuts the pie into $a + b$ pieces. (Equally.) Note that this take $a + b - 1$ cuts.

2. Bob takes b of the pieces (the biggest b pieces).

We leave it to the reader to show that if Alice does not cut the pieces equally she could do worse than $a/(a+b)$, and that if Alice follows the advice then Alice will get at least $a/(a + b)$ and Bob will get at least $b/(a + b)$. ∎

Can we do $(a : b)$ division with fewer cuts? YES, as we will see in the next section.

Note that Theorem 2.1 only used one ROUND Of cuts. We suspect that if you only make one round of cuts then $a + b - 1$ is optimal.

3 The Near-Halves 2-Player Protocol for Unfair Division

The key to this protocol is that we will reduce the problem by alot in each stage.

Theorem 3.1 *For all a, b there is a 2-player protocol for $(a : b)$ division that takes $\le \lceil \lg(a+b) \rceil$ cuts.*

Proof:

We assume $a \le b$. We also will DO a division of both sides by a common factor if it comes up. That is why this is a $\le$ instead of an $=$. (Getting an $=$ would be difficult.)

1. If $a = b$ then do cut and choose. Else goto the next step.
2. If $a + b$ is even then do the following
   (a) Alice cuts the cake into two pieces (equal).
   (b) Bob picks one of the pieces (the biggest).
   (c) Do $(a : b - \frac{a+b}{2}) = (a : \frac{b-a}{2})$ unfair division on whats left. (reduce the fraction).
3. Alice cuts the cake into two pieces (ratio is $\frac{a+b-1}{2} : \frac{a+b+1}{2}$.
4. Bob picks one of the pieces. (the one that is as advertised- either the left piece if its $\ge \frac{a+b-1}{2}$ or the right piece if its $\ge \frac{a+1+1}{2}$.)
5. If Bob picks the left piece then divide the rest of the cake via $(a : b - \frac{a+b-1}{2}) = (a : \frac{b-a+1}{2})$. If Bob picks the right piece then divide the rest of the cake via $(a : b - \frac{a+b+1}{2}) = (a : \frac{b-a-1}{2})$. Note that the worst case is the reduction to We can assume the worst case is when the problem reduces to $(a : b - \frac{a+b-1}{2}) = (a : \frac{b-a+1}{2})$.

Let $L(a, b)$ where $a \le b$ be a bound on the number of cuts this algorithm uses. Then note that

$$L(a,b) \le \begin{cases} L(a, \frac{b-a}{2}) \text{ if } a+b \text{ is even and } a \le \frac{b-a}{2} \\ L(\frac{b-a}{2}, a) \text{ if } a+b \text{ is even and } a > \frac{b-a}{2} \\ L(a, \frac{b-a+1}{2}) \text{ if } a+b \text{ is odd and } a \le \frac{b-a+1}{2} \\ L(\frac{b-a+1}{2}, a) \text{ if } a+b \text{ is odd and } a > \frac{b-a+1}{2} \end{cases} \tag{1}$$

One can use the recurrences, and induction, to show the bound.

$\blacksquare$

Can we do better still? This is unclear; however, we give some thoughts in the next section.

4 A Magic x!

Imagine that Alice and Bob want to cut the cake in ratio $(88 : 65)$. Near-halves would take $\lceil \log_{(} 88+65)\rceil = \lceil \log_2(153)\rceil = 8$ cuts. Can they do better? YES!

1. Alice cuts a piece worth 10 out of 153.
   (a) If Bob likes it, he takes it, and the problem is now $(88 : 55) = (8 : 5)$. By Near-halves $(8 : 5)$ can be done in $\lceil \log_2(13)\rceil = 4$. Hence total number of cuts is 5.
   (b) If Bob does not like it then Alice takes it and the problem is now $(78 : 65) = (6 : 5)$ By Near-halves $(6 : 5)$ can be done in $\lceil \log_2(11)\rceil = 4$. Hence total number of cuts is 5.

We got very very luck here. The cut of 10 made BOTH sides reduce ALOT because of divisibility. Can we always find such magic cuts? No. In the next section we discuss how to find the best algorithm if it exists. This algorithm will look at ALL cuts.

5 The Ultimate Algorithm

Can we always find the best algorithm? Yes, though it will take some work. We first describe it as a recurrence and then as a dynamic program.

Lets say Alice and Bob want to divide a cake in the ratio $(7 : 12)$. Look at Alice's options:

- Alice could cut it in ratio $1 : 18$. If Bob wants the (small!) piece he takes it and the problem is now $(7 : 11)$. If not then Alice takes that small piece and the problem is now $(6 : 18) = (1 : 3)$.

- Alice could cut it in ratio $2 : 17$. If Bob wants the (small!) piece he takes it and the problem is now $(7 : 10)$. If not then Alice takes that small piece and the problem is now $(5 : 18)$.

- Alice could cut it in ratio $3 : 16$. If Bob wants the (small!) piece he takes it and the problem is now $(7 : 9)$. If not then Alice takes that small piece and the problem is now $(4 : 18) = (2 : 9)$.

- Alice could cut it in ratio $4 : 15$. If Bob wants the (small!) piece he takes it and the problem is now $(7 : 8)$. If not then Alice takes that small piece and the problem is now $(3 : 18) = (1 : 6)$.

- Alice could cut it in ratio $5 : 14$. If Bob wants the (small!) piece he takes it and the problem is now $(7 : 7) = (1 : 1)$. If not then Alice takes that small piece and the problem is now $(2 : 18) = (1 : 9)$.

- Alice could cut it in ratio $6 : 13$. If Bob wants the (small!) piece he takes it and the problem is now $(7 : 6)$. If not then Alice takes that small piece and the problem is now $(1 : 18)$.

- Alice could cut it in ratio $7 : 12$. If Bob wants the (small!) piece he takes it and the problem is now $(7 : 5)$ If not then Alice takes that small piece and the problem is now $(0 : 12)$.

- Alice could cut it in ratio $8 : 11$. KEY: This is different. Now the piece that is available is too big for Alice. However, both pieces are okay for Bob. So he will pick one of them. If Bob wants the small piece he takes it and the problem is now $(7 : 4)$ If Bob wants the big piece he takes it and the problem is now $(7 : 1)$.

- Alice could cut it in ratio $9 : 10$. (This is the Near-halves protocol.) If Bob wants the small piece he takes it and the problem is now $(7 : 3)$ If Bob wants the big piece he takes it and the problem is now $(7 : 2)$.

We stopped here since the next case is to cut it in ratio $10 : 9$ and that is the same as $9 : 10$.

Which one is best? To determine that we would need to know all of the subproblems that arose. Hence we have the following recurrence: Let $OPT(a:b)$ be the optimal number of cuts for ratio $(a:b)$.

1. $OPT(1:1)=1$.
2. $OPT(1:b)=OPT(b:1)=\lceil \lg(1+b)\rceil$.
3. $OPT(a:b)$ is 1 PLUS the MIN of the following
   (a) MAX as $1 \le x \le a$ of $OPT(a-x:b)$ and $OPT(a:b-x)$. (This is when the piece cut is small so that if Bob declines it, Alice takes it.) (You might get $b-x<a$ so you would need to really recurse to $OPT(b-x:a)$ in the second case.)
   (b) MAX as $1 \le x, a+b-x \le b$ of $OPT(a:b-x)$ and $OPT(a:b-(a+b-x))$ (This is when the piece cut is big for Alice so that if Bob which piece to take.) (You might get $b-(a+b-x)$ so you would need to really recurse to $OPT(b-(a+b-x):a)$ in the second case.)

A recursive program could be written to solve this. This is a bad idea- there will be alot of redundant computing. Instead use a Dynamic program In any case you could store not just the value OPT but also the values x which will give you the algorithm.

Does this procedure give the BEST algorithm? Sadly no. The above assumes that all cuts are multiples of $\frac{1}{a+b}$. There are cases where that is not true. One such case is $(5:14)$ where the first cut in the optimal algorithm is to cut $(3:38)$.

6 Is Log(a+b) a Lower Bound

It was conjectured (by me), for all but a finite number of a,b, $OPT(a:b) \ge \Omega(\log_2(a+b))$. Andrew Lohr (goto arXiv for the latest version of his paper) showed that this is NOT true. He showed the following:

An example where the number of slices is really small is (58,470,565:72,019,008) which can be done in 6 steps. Note that these numbers are relatively prime. Note also that $\lceil \lg(58470565+72019008)\rceil = 27$, so the OPT algorithm is is a substantial improvement over the near-halves algorithm.

There are an infinite number of a, b such that $OPT(a:b) \le \log_2(\log_2(a+b))$.

He also showed that this is the best you can do:

For all but a finite number of a, b, $OPT(a:b) \ge \log_2(\log_2(a+b))$.

7 What if Alice and Bob have Linear Valuations?

In this section the implicit protocol is that Alice and Bob REVEAL their linear valuation functions an then find a value of x to cut where they are both happy.

We do an example.

We view the cake as the interval $[0,1]$.

Let Alice's tastes be determined by $v(a,b) = \int_a^b f(x)dx$ where $f(x) = x + \frac{1}{2}$. Note that $v(0,x) = \frac{x}{2}(x+1)$.

Let Bob's tastes be determined by $v(a,b) = \int_a^b g(x)dx$ where $g(x) = \frac{4x}{3} + \frac{1}{3}$. Note that $v(x,1) = 1 - \frac{x}{3}(2x+1)$.

Problem 1: Find an x such that if they cut at x, Alice getting the LEFT, Bob getting the RIGHT, Alice and Bob get the same value. Note what ratio THEY think they get.

Need

$$\begin{aligned} \tfrac{x}{2}(x+1) &= 1 - \tfrac{x}{3}(2x+1) \\ 3x(x+1) &= 6 - 2x(2x+1) \\ 3x^2 + 3x &= 6 - 4x^2 - 2x \\ 7x^2 + 5x - 6 &= 0 \end{aligned}$$

$$x = \frac{-5 \pm \sqrt{25 + 4 \times 7 \times 6}}{14} = \frac{-5 \pm \sqrt{193}}{14}$$

Only the + root makes sense for our purposes so

$$x = \frac{-5 \pm \sqrt{193}}{14} \sim 0.635$$

Alice gets $\frac{x}{2}(x+1) \sim .519$ Alice thinks that Alice/Bob $= .519/.481 = 1.07$.

Bob gets the same.

Problem 2: Find an x such that if they cut at x, Alice getting the RIGHT Bob getting the LEFT, Alice and Bob get the same value. Note what ratio THEY think they get.

Need

$$\begin{aligned} 1 - \tfrac{x}{2}(x+1) &= \tfrac{x}{3}(2x+1) \\ 6 - 3x(x+1) &= 2x(2x+1) \\ 6 - 3x^2 - 3x &= 4x^2 + 2x \\ 0 &= 7x^2 + 5x - 6 \end{aligned}$$

SAME equation not a coincidence. So get $x \sim 0.635$ But now
Alice and Bob both get .481, WORSE.
UPSHOT- it matters what side they get.

Problem 3: Find ALL x such that if they cut at x, Alice getting the LEFT, Bob getting the RIGHT, Alice thinks she has at least 1/3 and Bob thinks he has at least 2/3 (so the ratio is (1 : 2)).

Alice gets at least 1/3:

$$\begin{aligned} \tfrac{x}{2}(x+1) &\geq 1/3 \\ 3x(x+1) &\geq 2 \\ 3x^2 + 3x &\geq 2 \\ 3x^2 + 3x - 2 &\geq 0 \end{aligned}$$

The bigger x is the better for Alice so we will find a root and demand that the cut be at least as big as the root. We will need the plus root.

$$x \geq \frac{-3 + \sqrt{9 + 4 \times 3 \times 2}}{6} = \frac{-3 + \sqrt{33}}{6} \sim 0.457$$

Bob gets at least 2/3:

$$\begin{aligned} 1 - \tfrac{x}{3}(2x+1) &\geq 2/3 \\ 6 - 2x(2x+1) &\geq 4 \\ 2 - 4x^2 - 2x &\geq 0 \\ 4x^2 + 2x - 2 &\leq 0 \end{aligned}$$

The smaller x is the better for Bob so we will find a root and demand that the cut be at most as big as the root. We will need the plus root.

$$x \leq \frac{-2 + \sqrt{4 + 4 \times 4 \times 2}}{8} = \frac{-2 + \sqrt{36}}{8} = 0.5$$

SO, if

$$0.457 \leq x \leq 0.5$$

then Bob will think he has at least 1/3 and Alice will think she has at least 2/3.

Problem 4: Find an x such that if they cut at x, Alice getting the LEFT, Bob getting the RIGHT, Alice thinks she has at least 1/3 and Bob thinks he has at least 2/3 (so the ratio is $(1:2)$), AND Alice + Bob is maximized.

From Problem 3 we know that

$$0.457 \le x \le 0.5$$

If you use calculus you will find that the max occurs at one of the end-points.

If $x = 0.457$ then

- Alice gets $\frac{0.457}{2}(1.457) = 0.333$.
- Bob gets $1 - (\frac{0.457}{3}(2 \times 0.457 + 1)) = 0.708$.
- Total: 1.041.

If $x = 0.5$ then

- Alice gets $\frac{0.5}{2}(1.5) = 0.375$
- Bob gets $1 - (\frac{0.5}{3}(2 \times 0.5 + 1)) = 1 - \frac{1}{3} = 0.66$.
- Total: 0.96.

SO better off with 0.457.

8 3-Player protocol for Unfair Division

Alice, Bob, and Carol want to split the cake $(a : b : c)$.

1. Alice and Bob split the cake in ratio $(a : b)$. So FOR NOW Alice has $\frac{a}{a+b}$ and Bob has $\frac{b}{a+b}$.
2. Alice and Carol split the part Alice got in ratio $(a + b : c)$. They are spliting $\frac{a}{a+b}$ cake. Alice gets

$$\frac{a+b}{a+b+c} \times \frac{a}{a+b} = \frac{a}{a+b+c}.$$

Carol ges

$$\frac{c}{a+b+c} \times \frac{a}{a+b}.$$

3. Bob and Carol split the part Bob got in ratio $(a+b : c)$. They are spliting $\frac{a}{a+b}$ cake. Bob gets

$$\frac{a+b}{a+b+c} \times \frac{b}{a+b} = \frac{b}{a+b+c}.$$

Carol ges

$$\frac{c}{a+b+c} \times \frac{b}{a+b}.$$

Alice has $\frac{a}{a+b} \times \frac{a+b}{a+b+c} = \frac{a}{a+b+c}$.
Bob has $\frac{b}{a+b} \times \frac{a+b}{a+b+c} = \frac{b}{a+b+c}$.
Carol has $\frac{a}{a+b} \times \frac{c}{a+b+c} + \frac{b}{a+b} \times \frac{c}{a+b+c} = (\frac{a}{a+b} + \frac{b}{a+b}) \times \frac{c}{a+b+c} = \frac{c}{a+b+c}$
How many cuts did this take?
$CUTS(a,b,c) = CUTS(a,a+b) + 2CUTS(a+b,c) \sim log(2a+b) + 2log(a+b+c)$