Examples of Three Person Cake Cutting With Uniform Valuations

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The paper

How to Cut a Cake Before the Party Ends
by
David Kurokawa, John K. Lai, Ariel Procaccia

has a protocol for envy-free cake cutting with piecewise linear valuations. Their paper inspired these slides. We refer to their paper as ENDS.
Alice’s tastes are uniform on $[\frac{1}{8}, 1]$. Multiplier: $\frac{8}{7}$.

Bob’s tastes are uniform on $[0, \frac{2}{3}]$. Multiplier: $\frac{3}{2}$.

Carol’s tastes are uniform on $[\frac{1}{4}, \frac{3}{4}]$. Multiplier: 2.
Intervals

How much of $I_1$ should Alice get?
How much of $I_1$ should Bob get?
How much of $I_1$ should Carol get?
How much of $I_2$ should Alice get?
How much of $I_2$ should Bob get?
How much of $I_2$ should Carol get?
Etc.
Variables

\(x_{1A}\) is how much Alice gets of \(I_1\).
\(x_{1B}\) is how much Bob gets of \(I_1\).
\(x_{1C}\) is how much Carol gets of \(I_1\).
\(x_{2A}\) is how much Alice gets of \(I_2\).
\(x_{2B}\) is how much Bob gets of \(I_2\).
\(x_{2C}\) is how much Carol gets of \(I_2\).
\vdots
\(x_{iP}\) is how much Person P gets of \(I_i\).

NOTE: \(x_{1A} = x_{4B} = x_{5B} = x_{1C} = x_{2C} = x_{5C} = 0\).

Example: \(x_{2A} = \frac{1}{10}\) → Alice gets subinterval of \(I_2\) of length \(\frac{1}{10}\).
Equations: The $x_i^P$ Make Sense

$l_1$ of length $\frac{1}{8}$: $0 \leq x_{1B} \leq \frac{1}{8} - 0 = \frac{1}{8}$

$l_2$ of length $\frac{1}{8}$: $0 \leq x_{2A}, x_{2B} \leq \frac{1}{4} - \frac{1}{8} = \frac{1}{8}$.

$l_3$ of length $\frac{5}{12}$: $0 \leq x_{3A}, x_{3B}, x_{3C} \leq \frac{2}{3} - \frac{1}{4} = \frac{5}{12}$

$l_4$ of length $\frac{1}{12}$: $0 \leq x_{4A}, x_{4C} \leq \frac{3}{4} - \frac{2}{3} = \frac{1}{12}$

$l_5$ of length $\frac{1}{4}$: $0 \leq x_{5A} \leq 1 - \frac{3}{4} = \frac{1}{4}$

We will not mention these again for a while.
Equations: The $x_{ip}$ Make Sense

$l_1$ of length $\frac{1}{8}$: $x_1B = \frac{1}{8}$

$l_2$ of length $\frac{1}{8}$: $x_2A + x_2B = \frac{1}{8}$

$l_3$ of length $\frac{5}{12}$: $x_3A + x_3B + x_3C = \frac{5}{12}$

$l_4$ of length $\frac{1}{12}$: $x_4A + x_4C = \frac{1}{12}$

$l_5$ of length $\frac{1}{4}$: $x_5A = \frac{1}{4}$

We set

$x_1B = \frac{1}{8}, \quad x_5A = \frac{1}{4}$.

The first and fifth equation are now satisfied.
Equations: Getting Everyone $\geq \frac{1}{3}$

Alice gets $\geq \frac{1}{3}$: $\frac{8}{7}(x_2A + x_3A + x_4A + \frac{1}{4}) \geq \frac{1}{3}$

$$\frac{8}{7}(x_2A + x_3A + x_4A) \geq \frac{1}{21}$$

Bob gets $\geq \frac{1}{3}$: $\frac{3}{2}(\frac{1}{8} + x_2B + x_3B) \geq \frac{1}{3}$

$$\frac{3}{2}(x_2B + x_3B) \geq \frac{7}{48}$$

Carol gets $\geq \frac{1}{3}$:

$$2(x_3C + x_4C) \geq \frac{1}{3}$$
All the Equations

All vars $\geq 0$.

\[ x_{2A} + x_{2B} = \frac{1}{8} \]

\[ x_{3A} + x_{3B} + x_{3C} = \frac{5}{12} \]

\[ x_{4A} + x_{4C} = \frac{1}{12} \]

\[ \frac{8}{7} (x_{2A} + x_{3A} + x_{4A}) \geq \frac{1}{21} \]

\[ \frac{3}{2} (x_{2B} + x_{3B}) \geq \frac{7}{48} \]

\[ 2 (x_{3C} + x_{4C}) \geq \frac{1}{3} \]

Can solve by REASONING or by an LP package.
**Reasoning: Carol First**

- Give Carol first – she has largest multiplier.
- Give Carol from $I_4$, only Alice competes there.
- Give her ALL of $I_4$ since still does not get Carol $\frac{1}{3}$.
- Recall:
  \[
  x_{4A} + x_{4C} = \frac{1}{12}
  \]
  \[
  2(x_{3C} + x_{4C}) \geq \frac{1}{3}
  \]
- Set $x_{4C} = \frac{1}{12}$. Forces $x_{4A} = 0$.
- $2(x_{3C} + \frac{1}{12}) \geq \frac{1}{3}$
- Set $x_{3C} = \frac{1}{6} - \frac{1}{12} = \frac{1}{12}$.
- Carol has $1/3$, Interval $I_4$ is allocated.
Plugging in $x_{4A} = 0$, $x_{4C} = \frac{1}{12}$, $x_{3C} = \frac{1}{12}$ yields:

$$x_{2A} + x_{2B} = \frac{1}{8}$$

$$x_{3A} + x_{3B} = \frac{1}{3}$$

$$\frac{8}{7}(x_{2A} + x_{3A}) \geq \frac{1}{21}$$

$$\frac{3}{2}(x_{2B} + x_{3B}) \geq \frac{7}{48}$$

**Satisfy Bob:** Give Bob from smaller interval $l_2$ (makes math easier) give him ALL of it: $x_{2B} = \frac{1}{8}$. Forces $x_{2A} = 0$. 
Making Bob Happy

Plug in $x_{2B} = \frac{1}{8}$ and $x_{2A} = 0$.

\[ x_{3A} + x_{3B} = \frac{1}{3} \]
\[ \frac{8}{7}(x_{3A}) \geq \frac{1}{21} \]
\[ \frac{3}{2}(\frac{1}{8} + x_{3B}) \geq \frac{7}{48} \]

Give Bob enough of $l_2$ so that he is happy:

\[ \frac{1}{8} + x_{3B} \geq \frac{7}{72} \]
\[ x_{3B} \geq \frac{55}{576} \]

Set $x_{3B} = \frac{55}{576}$. Forces $x_{3A} = \frac{1}{3} - \frac{55}{576} = \frac{137}{576}$. Does this work?
Alice: $x_1A = 0$, $x_2A = 0$, $x_3A = \frac{137}{576}$, $x_4A = 0$, $x_5A = \frac{1}{4}$.

$$\frac{8}{7}(0 + 0 + \frac{137}{576} + 0 + \frac{1}{4}) \sim 0.5575$$

Bob: $x_1B = \frac{1}{8}$, $x_2B = \frac{1}{8}$, $x_3B = \frac{55}{576}$, $x_4B = 0$, $x_5B = 0$.

$$\frac{3}{2}(\frac{1}{8} + 0 + \frac{1}{8} + \frac{55}{576} + 0 + 0) \sim 0.5182$$

Carol: $x_1C = 0$, $x_2C = 0$, $x_3C = \frac{1}{12}$, $x_4C = \frac{1}{12}$, $x_5C = 0$.

$$2(0 + 0 + \frac{1}{12} + \frac{1}{12} + 0) = \frac{1}{3} \sim 0.3333$$

TOTAL:

$$0.5575 + 0.5182 + 0.3333 = 1.409$$

MOST UNHAPPY: Carol with 0.33333.
The Linear Programming Problem Maximize (or Minimize) a LINEAR function relative to LINEAR constraints.

Example
Maximize

\[ 4x + 8y - 7z \]

Relative to

\[-3x + 5y - 8z \leq 20\]
\[ x + y + z \leq 5\]
\[ 2x + y + 18z \leq 100\]
\[ 7x + 29y + 178z \leq 193\]

▶ VERY practical problem. Many REAL applications.
▶ There are MANY PACKAGE for it that are easy to use:
  http://www3.nd.edu/~jeff/mathprog/mathprog.html
We want $x_2A, x_2B, x_3A, x_3B, x_3C, x_4A, x_4C$ that satisfies:

$0 \leq x_2A, x_2B \leq \frac{1}{8}$

$0 \leq x_3A, x_3B, x_3C \leq \frac{5}{12}$

$0 \leq x_4A, x_4C \leq \frac{1}{12}$

$x_2A + x_2B = \frac{1}{8}$

$x_3A + x_3B + x_3C = \frac{5}{12}$

$x_4A + x_4C = \frac{1}{12}$

$\frac{8}{7}(x_2A + x_3A + x_4A + \frac{1}{4}) \geq \frac{1}{3}$

$\frac{3}{2}(\frac{1}{8} + x_2B + x_3B) \geq \frac{1}{3}$

$2(x_3C + x_4C) \geq \frac{1}{3}$
What to Maximize?- TOTAL Happiness

Our Goal is WEAKER than Linear Programming- all we want to do is find SOME point.
But can use this framework:
MAXIMIZE total happiness
or
MINIMIZE individual unhappiness

\[
\frac{8}{7}(x_{2A} + x_{3A} + x_{4A} + \frac{1}{4}) + \frac{3}{2}(\frac{1}{8} + x_{2B} + x_{3B}) + 2(x_{3C} + x_{4C})
\]
Maximizing Total Happiness

Plugged into an LP package:
A: $x_{1A} = 0$, $x_{2A} = 0.0277$, $x_{3A} = 0.0138$, $x_{4A} = 0$. $x_{5A} = 0.25$

$$\frac{8}{7}(0 + 0.0277 + 0.0138 + 0 + 0.25) = 0.333$$

B: $x_{1B} = 0.125$, $x_{2B} = 0.0972$, $x_{3B} = 0$, $x_{4B} = 0$, $x_{5B} = 0$.

$$\frac{3}{2}(0.125 + 0.0972 + 0 + 0 = 0) = 0.333$$

C: $x_{1C} = 0$, $x_{2C} = 0$, $x_{3C} = 0.403$, $x_{4C} = 0.083$, $x_{5C} = 0$.

$$2(0 + 0 + 0.403 + 0.083 + 0) = 0.972$$

TOTAL:

$$0.3333 + 0.3333 + 0.97222 = 1.638$$

MOST UNHAPPY: Alice and Bob 0.3333.
Add a variable $t$.

\[
\frac{8}{7}(x_{2A} + x_{3A} + x_{4A} + \frac{1}{4}) \geq t
\]

\[
\frac{3}{2}(\frac{1}{8} + x_{2B} + x_{3B}) \geq t
\]

\[
2(x_{3C} + x_{4C}) \geq t
\]

Maximize $t$. 
Minimizing Ind. Unhappiness

Plugged into an LP package:
A: \( x_1A = 0, \ x_2A = 0, \ x_3A = 0.17857, \ x_4A = 0. \ x_5A = 0.25 \)

\[
\frac{8}{7}(0 + 0 + .178587 + 0.25) = 0.4898
\]

B: \( x_1B = 0.125, \ x_2B = 0.125, \ x_3B = 0.076531, \ x_4B = 0, \ x_5B = 0. \)

\[
\frac{3}{2}(0.125 + 0.125 + 0.076531 + 0 + 0) = 0.4898
\]

C: \( x_1C = 0, \ x_2C = 0, \ x_3C = 0.16156, \ x_4C = 0.083, \ x_5C = 0. \)

\[
2(0 + 0 + 0.16156 + 0.083 + 0) = 0.4898.
\]

TOTAL:

\[
0.4898 + 0.4898 + 0.4898 = 1.4694
\]

MOST UNHAPPY: ALL have 0.4898.
Protocol for $n$ players, all have uniform valuations.

1. Every player simul reveals their valuation. (honestly)
2. Players form LP program to satisfy that all have $\geq 1/n$, vars make sense, and total is maximized (OR to minimize Unhappiness). They solve the LP.
3. Player make the cuts as the LP solution dictates.

▶ How many cuts? $\leq 2n - 1$ intervals, $\leq n - 1$ cuts. PLUS the cuts at each interval, $\leq 2n - 2$ cuts. TOTAL NUMBER OF CUTS: $\leq (2n - 1)(n - 1) + 2n - 2 = 2n^2 - n - 2$.

▶ Does this LP always have a solution? Yes.

▶ The paper ENDS has an $O(n^2)$ protocol for envy-free (hence prop) but does not maximize total. Extends to piece-wise valuations but with diff bound depending on number-of-pieces.
Inequalities for Envy Free:

Alice not envious of Bob: \( x_{2A} + x_{3A} + x_{4A} + \frac{1}{4} \geq x_{2B} + x_{3B} \).

Alice not envious of Carol: \( x_{2A} + x_{3A} + x_{4A} + \frac{1}{4} \geq x_{3C} + x_{4C} \).

Bob not envious of Alice: \( \frac{1}{8} + x_{2B} + x_{3B} \geq x_{2A} + x_{3A} \).

Bob not envious of Carol: \( \frac{1}{8} + x_{2B} + x_{3B} \geq x_{3C} \).

Carol not envious of Alice: \( x_{3C} + x_{4C} \geq x_{3A} + x_{4A} \).

Carol not envious of Bob: \( x_{3C} + x_{4C} \geq x_{3B} \).
All Constraints for Envy Free

\[ x_2A + x_2B = \frac{1}{8} \]

\[ x_3A + x_3B + x_3C = \frac{5}{12} \]

\[ x_4A + x_4C = \frac{1}{12} \]

\[ x_2A + x_3A + x_4A + \frac{1}{4} \geq x_2B + x_3B \]

\[ x_2A + x_3A + x_4A + \frac{1}{4} \geq x_3C + x_4C \]

\[ \frac{1}{8} + x_2B + x_3B \geq x_2A + x_3A \]

\[ \frac{1}{8} + x_2B + x_3B \geq x_3C \]

\[ x_3C + x_4C \geq x_3A + x_4A \]

\[ x_3C + x_4C \geq x_3B \]
Maximize Total:
Alice: $x_{1A} = 0$, $x_{2A} = 0$, $x_{3A} = 0.1111$, $x_{4A} = 0$, $x_{5A} = 0.25$.

$$\frac{8}{7}(0 + 0 + 0.1111 + 0 + 0 + 0.25) \sim 0.4126$$

Bob: $x_{1B} = 0.125$, $x_{2B} = 0.125$, $x_{3B} = 0.02777$, $x_{4B} = 0$, $x_{5B} = 0$.

$$\frac{3}{2}(0.125 + 0.125 + 0.02778 + 0 + 0) \sim 0.41667$$

Carol: $x_{1C} = 0$, $x_{2C} = 0$, $x_{3C} = 0.2777$, $x_{4C} = 0.08333$, $x_{5C} = 0$.

$$2(0 + 0 + 0.2777 + 0.08333) \sim 0.722$$

TOTAL:

$$0.4162 + 0.4166 + 0.722 = 1.5512$$

MOST UNHAPPY: Alice with 0.4126.
Got same numbers as wanted just proportional and min unhappiness.
Envy Free Protocol for $n$ players, all have uniform valuations.

1. Every player simul reveals their valuation. (honestly)
2. Players form LP program to satisfy that there is no envy, all vars make sense, and total is maximized. (They set the obv vars to 0 and whatever else is forced.) They solve the LP.
3. Player make the cuts as the LP solution dictates.

- How many cuts? As before $\leq 2n^2 - n - 2$.
- Does this LP always have a solution? Yes.
- The paper ENDS has an $O(n^2)$ protocol for envy-free (hence prop) but does not maximize total. Extends to piece-wise valuations but with diff bound depending on number-of-pieces.
Other Valuations

What if Valuation is of
\[ v(c, d) = \int_c^d (ax + b) \, dx = \frac{a}{2} (d^2 - c^2) + b(d - c). \]

Only makes sense if \( 1 = v(0, 1) = \int_0^1 (ax + b) \, dx = \frac{a}{2} + b. \)

\[ 1 = \frac{a}{2} + b \]

We do an example.
Let $f(x) = 2x$, $g(x) = x + \frac{1}{2}$, $h(x) = \frac{x}{2} + \frac{3}{4}$.

Alice’s Val: $val_A(b, a) = \int_a^b f(x) = b^2 - a^2$.

Bob’s Val: $val_B(b, a) = \int_a^b g(x) = \frac{1}{2}(b^2 - a^2) + \frac{1}{2}(b - a)$.

Carol’s Val: $val_C(b, a) = \int_a^b h(x) = \frac{1}{4}(b^2 - a^2) + \frac{3}{4}(b - a)$.

**Note:** $f(x), g(x), h(x)$ all MEET at $(\frac{1}{2}, 1)$. 
This is DIFF than before.

\[ 0 \rightarrow x_1 \rightarrow x_2 \rightarrow \frac{1}{2} \rightarrow x_3 \rightarrow x_4 \rightarrow 1 \]

- A gets \([x_2, \frac{1}{2}] \cup [\frac{1}{2}, x_3]\)
- B gets \([x_1, x_2] \cup [x_3, x_4]\)
- C gets \([0, x_1] \cup [x_4, 1]\)
A gets

\[ \left( \frac{1}{2} \right)^2 - x_2^2 + x_3^2 - \left( \frac{1}{2} \right)^2 = x_3^2 - x_2^2 \]

B gets

\[ \frac{1}{2}(x_2^2 - x_1^2 + x_4^2 - x_3^2) + \frac{1}{2}(x_2 - x_1 + x_4 - x_3) \]

C gets

\[ \frac{1}{4}(x_1^2 + 1 - x_4^2) + \frac{3}{4}(x_1 + 1 - x_4) \]
Alice’s View of the World

Alice thinks:
Alice gets $x_3^2 - x_2^2$
Bob gets $x_2^2 - x_1^2 + x_4^2 - x_3^2$
Carol gets $x_1^2 + 1 - x_4^2$.

Equations so that Alice has no envy:
$x_3^2 - x_2^2 \geq x_2^2 - x_1^2 + x_4^2 - x_3^2$
$x_3^2 - x_2^2 \geq x_1^2 + 1 - x_4^2$. 
Bob thinks:
Alice gets $\frac{1}{2}(x_3^2 - x_2^2) + \frac{1}{2}(x_3 - x_2)$
Bob gets $\frac{1}{2}(x_2^2 - x_1^2 + x_4^2 - x_3^2) + \frac{1}{2}(x_2 - x_1 + x_4 - x_3)$
Carl gets $\frac{1}{2}(x_1^2 + 1 - x_4^2) + \frac{1}{2}(x_1 + 1 - x_4)$

Equations so that Bob has no envy:
$(x_2^2 - x_1^2 + x_4^2 - x_3^2) + (x_2 - x_1 + x_4 - x_3) \geq (x_3^2 - x_2^2) + (x_3 - x_2)$
$(x_2^2 - x_1^2 + x_4^2 - x_3^2) + (x_2 - x_1 + x_4 - x_3) \geq (x_1^2 + 1 - x_4^2) + (x_1 + 1 - x_4)$
Carol’s View of the World

Carol thinks:
Alice gets $\frac{3}{4}(x_3^2 - x_2^2) + \frac{1}{4}(x_3 - x_2)$
Bob gets $\frac{3}{4}(x_2^2 - x_1^2 + x_4^2 - x_3^2) + \frac{1}{4}(x_2 - x_1 + x_4 - x_3)$
Carol gets $\frac{3}{4}(x_1^2 + 1 - x_4^2) + \frac{1}{4}(x_1 + 1 - x_4)$

Equations so that Bob has no envy:
$3(x_1^2 + 1 - x_4^2) + (x_1 + 1 - x_4) \geq 3(x_3^2 - x_2^2) + (x_3 - x_2)$
$3(x_1^2 + 1 - x_4^2) + (x_1 + 1 - x_4) \geq 3(x_2^2 - x_1^2 + x_4^2 - x_3^2) + (x_2 - x_1 + x_4 - x_3)$
Problem 1: Does there exist \( x_1, x_2, x_3, x_4 \) that satisfies the following inequalities:

\[
0 \leq x_1 \leq x_2 \leq x_3 \leq x_4 \leq 1
\]

\[
x_3^2 - x_2^2 \geq x_2^2 - x_1^2 + x_4^2 - x_3^2
\]

\[
x_3^2 - x_2^2 \geq x_1^2 + 1 - x_4^2.
\]

\[
(x_2^2 - x_1^2 + x_4^2 - x_3^2) + (x_2 - x_1 + x_4 - x_3) \geq (x_3^2 - x_2^2) + (x_3 - x_2)
\]

\[
(x_2^2 - x_1^2 + x_4^2 - x_3^2) + (x_2 - x_1 + x_4 - x_3) \geq (x_1^2 + 1 - x_4^2) + (x_1 + 1 - x_4)
\]

\[
3(x_1^2 + 1 - x_4^2) + (x_1 + 1 - x_4) \geq 3(x_3^2 - x_2^2) + (x_3 - x_2)
\]

\[
3(x_1^2 + 1 - x_4^2) + (x_1 + 1 - x_4) \geq 3(x_2^2 - x_1^2 + x_4^2 - x_3^2) + (x_2 - x_1 + x_4 - x_3)
\]

Note: Can Phrase as Quad Prog Problem.
The Quadratic Programming Problem  Maximize (or Minimize)  
a LINEAR function relative to QUADRATIC constraints.  

Example  
Maximize  

$$4x + 8y - 7z$$  

Relative to  

$$-3x^2 + 5y - 8z^2 \leq 20$$  
$$x^2 + y^2 + z \leq 5$$  
$$2x + y^2 + 18z \leq 100$$  
$$7x + 29y + 178z^2 \leq 193$$  

▶ NP-Hard. Thought to be HARD.  
▶ There is ONE PACKAGES for it that I know.
Problem 2: Maximize

\[
\left(\frac{1}{2}\right)^2 - x_2^2 + x_3^2 - \left(\frac{1}{2}\right)^2 + x_3^2 - x_2^2 + \frac{1}{2}(x_2^2 - x_1^2 + x_4^2 - x_3^2) + \frac{1}{2}(x_2 - x_1 + x_4 - x_3) + \frac{1}{4}(x_1^2 + 1 - x_4^2) + \frac{3}{4}(x_1 + 1 - x_4)
\]

while satisfying:

0 \leq x_1 \leq x_2 \leq x_3 \leq x_4 \leq 1

x_3^2 - x_2^2 \geq x_2^2 - x_1^2 + x_4^2 - x_3^2
\]

x_3^2 - x_2^2 \geq x_1^2 + 1 - x_4^2.

(x_2^2 - x_1^2 + x_4^2 - x_3^2) + (x_2 - x_1 + x_4 - x_3) \geq (x_3^2 - x_2^2) + (x_3 - x_2)

(x_2^2 - x_1^2 + x_4^2 - x_3^2) + (x_2 - x_1 + x_4 - x_3) \geq (x_1^2 + 1 - x_4^2) + (x_1 + 1 - x_4)

3(x_1^2 + 1 - x_4^2) + (x_1 + 1 - x_4) \geq 3(x_3^2 - x_2^2) + (x_3 - x_2)

3(x_1^2 + 1 - x_4^2) + (x_1 + 1 - x_4) \geq 3(x_2^2 - x_1^2 + x_4^2 - x_3^2) + (x_2 - x_1 + x_4 - x_3)
We want to maximize a Quadratic function relative to Quadratic Constraints. We call this Quadratic Quadratic Programming (QQP). QQP has not been studied. Rumors of a packages that might solve it.

SOOL? FML? FUBAR? FML!!! My prof wants me to solve a QQP!!!
Envy Free Protocol for $n$ players, all have linear valuations.

1. Every player simul reveals their valuation. (honestly)
2. Players form QQP program to satisfy that there is no envy, all vars make sense, and total is maximized. Solve the QQP.
3. If someone starves to death while solving the QQP then remove them and re-do equations. Repeat if needed.
4. If there are $\geq 2$ people left when solved then use the solution. If there is only 1 person left, he gets it.
Serious Protocol and Open Questions

Envy Free Protocol for $n$ players, all have linear valuations.

1. Every player simul reveals their valuation. (honestly)
2. Players form QQP program to satisfy that there is no envy, all vars make sense, and total is maximized. Solve the QQP.
3. Solve it.
4. Cut the cake as it dictates.

▶ Does a QQP of his form always have a solution?
▶ Is there always a rational point that satisfies the constraints? Unlikely.
▶ Is there an efficient algorithm to find an approx solution to the QQP that arise from this problem? (Do not know?)
▶ Will these be solved before or after the Gov. Shutdown ends?