Cut and Choose (cc)

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Credit Where Credit is Due

This goes back to the Bible

An Early Case: Abraham and Lot

In the bible

1. Abraham says to Lot
   Do you want the West or East Part of the Land (Cutting).
2. Lot says
   I'll take the Easter Part (Choosing).

For this talk

For this talk **Protocol** always means:

1. Two player.
2. Dividing a cake (a continuous good).
3. They may have different tastes (geometry not helpful).

General Procedure

Alice and Bob want to divide a cake

1. Alice cuts the cake in half (equal in her eyes)
2. Bob picks one of those pieces (the bigger one in his eyes)

Cheat Proof

Theorem

If Alice cheats then she might end up with LESS THAN she would have gotten if she had been honest.

Proof.

Scenario: Alice cuts the cake in P_1, P_2 where $V_A(P_1) < \frac{1}{2}$ and $V_A(P_2) > \frac{1}{2}$. Bob takes P_2–Alice has P_1 and $< \frac{1}{2}$ $\square$

Theorem

If Bob cheats then he might end up with LESS THAN she would have gotten had he been honest.

Theorem

The protocol is cheat-proof.

Will assume from now on that both players are honest.

Proportional and Envy Free

Theorem

The protocol is proportional and hence Envy Free.

Proof.

Alice thinks $V_A(P_1) = V_A(P_2) = \frac{1}{2}$.
So she always gets a piece of value $\geq 1/2$.
Bob will pick the bigger piece so he will get $\geq \frac{1}{2}$. $\square$

DISCUSSION

DISCUSS PROS AND CONS OF PROTOCOL

PRO

PRO

1. Proportional, Envy Free, Cheat proof.
2. Players need not have precise valuation.
3. Works for ANY valuations.
4. Pieces are continuous.

CON

CON

1. Not Equitable (Homework)
2. Alice is at a disadvantage (Homework)

ϵ-Equitable

Definition

A division (P_1, P_2) where Alice gets P_1 and Bob gets P_2 is *ϵ-Equitable* if

$$|V_A(P_1) - V_B(P_2)| < \epsilon.$$

What Do You Think?

Which of the following is true?

1. For all ϵ there exists an ϵ-equitable protocol.
2. There is ϵ such that there is no ϵ-equitable protocol.
3. The question is unknown to science.

What Do You Think?

Which of the following is true?

1. For all ϵ there exists an ϵ-equitable protocol.
2. There is ϵ such that there is no ϵ-equitable protocol.
3. The question is unknown to science.
4. Unknown until Feb 4, 2015, 11:00AM. Now known:
   There IS such a protocol!

Theorem and Proof

Theorem

For all ϵ there exists an ϵ-equitable protocol.

Proof.

The cake is the line $[0, 1]$.

1. Alice and Bob simul say a number. Alice says a Bob says b. (Alice thinks $V_A([0, a]) = V_A([a, 1])$ and Bob thinks $\ldots$). If $a \le b$ then Alice gets $[0, a)$, Bob gets $(b, 1]$. If $a > b$ then Alice gets $(a, 1]$ and Bob gets $[0, b)$. They both have $1/2$. We assume $a \le b$, other case similar.
2. If $a = b$ then DONE. Assume not. Have $[a, b]$ to split.
3. $V_A([a, b]) < \epsilon$ & $V_B([a, b]) < \epsilon$: cc on $[a, b]$-DONE.
4. $V_A([a, b]) \ge \epsilon$ or $V_B([a, b]) \ge \epsilon$: repeat with $[a, b]$.

$\square$

Theorem and Proof: More Elegant

Theorem

For all ϵ there exists an ϵ-equitable protocol.

Proof.

1. Input is (x, y). They will be dividing $[x, y]$.
2. Alice and Bob simul say a number. Alice says a Bob says b.
   (Alice thinks $V_A([x, a]) = V_A([a, y])$ and Bob thinks $\ldots$).
   If $a \le b$ then Alice gets $[x, a)$, Bob gets $(b, y]$.
   If $a > b$ then Alice gets $(a, y]$ and Bob gets $[x, b)$.
   They both have $1/2$ of $[x, y]$. We assume $a \le b$.
3. If $a = b$ then DONE. Assume not. Have $[a, b]$ to split.
4. $V_A([a, b]) < \epsilon$ & $V_B([a, b]) < \epsilon$: cc on $[a, b]$-DONE.
5. $V_A([a, b]) \ge \epsilon$ or $V_B([a, b]) \ge \epsilon$: call **RECURSIVELY** on (a, b).

$\square$

Why Equitable?

Let Alice and Bob execute the protocol. Let the sequence of Alice-cuts be $a_1 < a_2 < \cdots < a_n$ and the sequence of Bob-cuts be $b_1 > b_2 > \cdots > b_n$.

1. $V_A([0, a_1]) = V_B[b_1, 1])$.
2. $V_A([0, a_2]) = V_B[b_2, 1])$.
3. $\vdots$
4. $V_A([0, a_n]) = V_B[b_n, 1])$.

Alice has $[0, a_n]$, Bob has $[b_n, 1]$.
$V_A([a_n, b_n]) < \epsilon$ and $V_B([a_n, b_n]) < \epsilon$.
No matter how $[a_n, b_n]$ is split, Alice and Bob will differ by $< \epsilon$.

Why Equitable?–I cheated on last slide

Let Alice and Bob execute the protocol. Let the sequence of Alice-cuts be $a_1 < a_2 < \cdots < a_n$ and the sequence of Bob-cuts be $b_1 > b_2 > \cdots > b_n$. **MIGHT NOT HAPPEN THAT WAY**.
Could be that $a_1 < b_1$ but $a_2 > b_2$. **Pieces might not be continuous**.
Let A_i be what Alice has after i iterations.
Let B_i be what Bob has after i iterations.

1. For all i, $V_A(A_i) = V_B(B_i)$.
2. $V_A(A_n) = V_B(B_n)$.

Alice has A_n, Bob has B_n. Only $[a_n, b_n]$ or $[1 - b_n, 1 - a_n]$ is unclaimed.
We assume $[a_n, b_n]$.
$V_A([a_n, b_n]) < \epsilon$ and $V_B([a_n, b_n]) < \epsilon$.
No matter how $[a_n, b_n]$ is split, Alice and Bob will differ by $< \epsilon$.

DISCUSSION

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PRO

PRO

1. Proportional, Envy Free, Cheat proof, ϵ-equitable.
2. Works for ANY valuations.
3. I came up with it!

CON

CON

1. Alice and Bob need to quantify their valuations.
2. Could take a long time.
3. Pieces not continuous.

Super Cheat Proof

Definition

A protocol is *super cheat proof* if even if you know your opponents tastes, cheating may lead to a worse outcome for you.

Obtaining this seems very hard. We may need to drop another requirement.

Definition

A protocol is *ϵ-proportional* each player has within ϵ of $\frac{1}{2}$.

ϵ-proportional Super Cheat Proof

Theorem

For all ϵ there exists an ϵ-proportional super-cheat-proof protocol. (Proven Friday Feb 6, 7:00PM).

Phase One

Let L be such that $\epsilon \leq \frac{1}{L}$.

1. Alice cuts into $2L$ pieces. (Evenly)
2. Bob cuts each piece into $2L$ pieces. (Evenly)
3. Alice and Bob reveal what each piece is worth.
4. Pieces: $p_1, \ldots, p_m$. $\mathrm{V}_A(p_i)$ be how much A values piece p_i. $\mathrm{V}_B(p_i)$ be how much B values piece p_i. (If both follow advice: $\mathrm{V}_A(p_i), \mathrm{V}_B(p_i) \leq \frac{1}{2L}$.)

Phase Two

Recall: All of the p_i's are TINY to both. Alice and Bob reorder the pieces.

1. $q_1 = p_1$.
2. Assume $q_1, \ldots, q_k$ are already defined.
   - If $\sum_{i=1}^{k} \mathrm{V}_A(q_i) \leq \sum_{i=1}^{k} \mathrm{V}_B(q_i)$ then Alice and Bob find a piece p not already used such that $\mathrm{V}_B(p) < \mathrm{V}_A(p)$. Let $q_{k+1} = p$.
   - If $\sum_{i=1}^{k} \mathrm{V}_B(q_i) < \sum_{i=1}^{k} \mathrm{V}_A(q_i)$ then Alice and Bob find a piece p not already used such that $\mathrm{V}_A(p) < \mathrm{V}_B(p)$. Let $q_{k+1} = p$.

Intuition: $q_1 \cup \cdots \cup q_k$ valued about the same by both.

Phase Three

$q_1, \ldots, q_n$ defined.
For all k, $q_1 \cup \cdots \cup q_k$ valued about the same to both.

1. Let k be the least number such that

$$\sum_{i=1}^{k} \mathrm{V}_A(q_i) \geq \frac{1}{2}.$$

2. Let $P = q_1 \cup \cdots \cup q_k$.
   Let $Q = q_{k+1} \cup \cdots \cup q_n$.
   Intuition: P and Q both valued about $\frac{1}{2}$ by both.
   Now what?

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   Let $Q = q_{k+1} \cup \cdots \cup q_n$.
   Intuition: P and Q both valued about $\frac{1}{2}$ by both.
   Now what?
3. FLIP A COIN!
   If its HEADS then Alice gets P, Bob gets Q.
   If its TAILS then Bob gets Q, Alice gets P.

ϵ-Proportional Super Cheat Proof

1. KEY: Neither player knows who will get P and who will get Q
2. BOTH want P and Q to be about the same size.
3. Neither will cheat for fear of getting the smaller piece.
4. Even if Alice knows Bob's tastes, no benefit to cheating.

DISCUSSION

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PRO

PRO

1. ϵ-Proportional, Super-Cheat proof
2. Works for ANY valuations.
3. I came up with it! (Based on things already known.)

CON

CON

1. ϵ-Proportional, not proportional.
2. **CRUMBS!**.
3. Alice and Bob need to be diamond cutters.

Is there a Proportional Super Cheat Proof Protocol?

VOTE

1. There is a proportional super cheat proof protocol.
2. There is a no proportional super cheat proof protocol.
3. The question is unknown to science.
4. The question was unknown to science until recently!

Its more complicated than that

The following are true:

1. There cannot be a protocol to create 2 pieces, size $\frac{1}{2}$.
2. Hence one approach to super-cheat proof is ruled out.
3. Open if it can be done.

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2. Hence one approach to super-cheat proof is ruled out.
3. Open if it can be done.
4. There is a protocol to create 2 pieces, size $\frac{1}{2}$.
5. Hence we can get super-cheat-proof.

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The following are true:

1. There cannot be a protocol to create 2 pieces, size $\frac{1}{2}$.
2. Hence one approach to super-cheat proof is ruled out.
3. Open if it can be done.
4. There is a protocol to create 2 pieces, size $\frac{1}{2}$.
5. Hence we can get super-cheat-proof.
6. What?

The rest on the board.