Proportional and Envy Free Moving Knife Divisions

1 Introduction

Whenever we say something like *Alice has a piece worth 1/2* we mean worth 1/2 TO HER.

Lets say we want Alice, Bob, Carol, to split a cake so that each one thinks they got at least 1/3. This is called a *proportional division*.

Lets say we want Alice, Bob, Carol, to split a cake so that each one thinks they got $\geq 1/3$ and that nobody else got a bigger piece. This is called an *envy-free division*.

This exposition is based on the articles by Brams, Taylor, and Zwicker [2, 3] and Barbanel and Taylor [1] If a reference is not given then see the survey of Brams, Taylor and Zwicker for the history.

2 An n-Person MK Proportional Protocol

The next theorem is due to Dubins and Spanier [4]

Theorem 2.1 *There is an n-person MK Proportional Protocol that takes only $n-1$ cuts.*

Proof:

If $n = 1$ then the one player just takes the case. Assume that $n \geq 2$ and that there IS an MK protocol for $n-1$.

1. A referee passes a knife over the cake until one of the n players yells STOP (someone will yell STOP when the knife is at the $1/n$ point).

2. Whoever yells gets that piece. (Break ties arbitrarily.)

3. Everyone else does the $n-1$ protocol with whats left.

If $n = 4$ then the knife moves until someone (say Alice) thinks its worth 1/4. So then the knife stops and Alice gets that piece. Note that Bob, Carol, and Donna thing its worth $\leq 1/4$. So they think whats left is $\geq 3/4$. It may be that (say) Bob thinks whats left is worth LOTS more than 3/4. They then use the $n = 3$ algorithm on the rest of the case. So the knife moves until one of them thinks that there is 1/3. Note that, with regard to the original cake, this is 1/3 of at least 3/4, so its at least 1/4 as it should be.

We need to show that if Alice is MOTIVATED to yell STOP when its at here $1/n$ point. If she yells BEFORE that point she gets LESS THAN $1/n$. If she (plans to yell) after that point then someone may yell first. That leaves her with splitting $< (n-1)/n$ with $n-1$ people so she gets $< 1/n$.

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3 A Lemma We will Need for Many Envy-Free Protocols

In the Lemma's in this section the protocol produces a piece or a division; however, nobody gets these pieces. If a player cheats then he will do worse in the protocols that use these lemma.

We will also use this lemma for a Theorem about 2-person cake cutting.

Lemma 3.1

1. *There is a 2-person MK protocol that will, given a cake, produce a division into two pieces that they both think are* $1/2$. *This takes 2 cuts.*

2. *There is a 2-person MK protocol that will, given a cake, produce a division into four pieces that they both think are all equal in size. This takes 6 cuts.*

Proof:
1)

1. Alice places one knife on the left end of the cake and the other such that the cake between the two is 1/2.

2. Alice moves the two knifes always keeping 1/2 between them.

3. Bob yells STOP when he thinks that the inside and outside are both 1/2.

4. They decide who gets inside vs outside with a coin toss.

If Bob first thought that the inside was $< 1/2$ and he outside was $> 1/2$ then there must be a point where he thinks they are equal (by either the intermediate value theorem or common sense).
2)

1. Apply the algorithm in part 1 to get two pieces that are 1/2 each to both of them.

2. Apply the algorithm in part 1 to both pieces.

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Recall that with Cut-and-choose the cutter might do much worse than the other person. We can use the above Lemma to have a protocol where they BOTh get 1/2.

Theorem 3.2 *There is a 2-person MK protocol where, at the end, they both have exactly 1/2.*

Proof:

1. Run the Protocol from Lemma 3.1.1 to produce two pieces P, Q that they both think are worth exactly 1/2.

2. Alice and Bob flip a coin to determine who gets P and who gets Q.

Since they do not know which person gets which piece they are both motiated to follow the advice of the protocol. ∎

4 3-Person MK Envy Free Protocols

Theorem 4.1 *There is a 3-Person MK Envy Free Protocol that takes 3 cuts.*

Proof:

1. The Knife moves and someone yells stop (when the piece is of size 1/3). The piece is cut (ONE CUT).

2. Say it was Alice who yelled. A coin is flipped to determine one of the players- Bob or Carol. We'll say its Bob. Alice and Bob do the protocol from Lemma 3.1 on the remainder of the cake (TWO CUTS). Call the pieces P, Q, R. Alice thinks $P = Q = R = 1/3$. Bob thinks $P \le Q = R$ and that $Q = R \ge 1/3$.

3. Carol takes (she is not envious since she got first pick), Bob takes (he is not envious since he thinks $Q = R$ are tied for first so he gets a big piece). Alice takes a piece (she is not envious since she thinks $P = Q = R$).

We need to prove that people are MOTIVATED to follow the advice.

If Alice yells to soon then she may end up with the small piece since she goes last.

If Alice yells to late, and still before Bob and Carol, again, she may end up with a small piece.

If Alice cheats on the Lemma 3.1 protocol then, again, she may end up with the small piece.

Bob is motivated to do the Lemma 3.1 protocol fairly since he may end up with a small piece (since he goes second).

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Here is another 3-person MK Envyfree Protocol that uses a CAKE (circular) instead of a pie, and it matters! (marginally).

Theorem 4.2 *There is a 3-Person MK Envy Free Protocol for CAKE cutting that takes 3 cuts.*

Proof:

1. Alice takes 3 knives and has each one go out from the center of the cake to the edge (so that each wedge is equal). She rotates them clockwise (keeping each wedge equal).

2. Someone yells (when they think that there are two wedges that are the same size and tied for first. This must happen by a continuity argument.)

3. Say Bob yells. Then Carol takes, Bob takes, Alice takes.

ISSUE- Why would anyone yell? I looked up the original paper and they don't address this.

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5 A 4-Person MK Envy Free Protocol

Brams, Taylor, and Zwicker [3] have proven the following.

Theorem 5.1 *There is a MK protocol for four people to achieve an envy-free division.*

Proof:

This algorithm is in two phases.

PHASE ONE:

1. Alice and Bob use the protocol from Lemma 3.1 to obtain four pieces they both think are worth exactly 1/4.

2. Carol trims a piece or not. (She wants to create a 2-way tie for best piece). The trimming is put aside for now.

3. Let the pieces be P_1, P_2, P_3, P_4. Let the trimming be T.

4. Donna picks a piece. (Biggest)

5. Carol picks a piece. If the trimmed piece was not already picked then she must pick it. (Biggest)

6. Bob picks a piece. (He thinks that both pieces left are the same size so no instructions needed.)

7. Alice takes the piece that is left.

We leave it as an exercise to show that they are all motivated to follow the advice and for Alice and Bob to execute the Lemma 3.1 protocol fairly.

Claim 1: The division of P_1, P_2, P_3, P_4 is envy free.

Proof:

Donna can't feel envy since she got to pick first.

Carol can't feel envy since she thought there was a tie for the best piece.

Alice and Bob can't feel envy since they thought that all untrimmed pieces were tied for best and each got an untrimmed piece.

End of Proof of Claim 1

We assume that Carol trimmed a piece and later got the trimmed piece (all the other cases are similar). In Phase II we deal with the trimming.

KEY: No matter how much Carol gets, neither Alice nor Bob can be envious of her since they just think she is getting back trim from a trimmed piece.

PHASE TWO:

1. Donna and Bob use the protocol from Lemma 3.1 to divide T into four pieces they both think are worth exactly 1/4.

2. Carol takes a piece.

3. Alice takes a piece.

4. Donna takes a piece.

5. Bob takes a piece.

Carol cannot feel envy since she picked first.

Alice is the interesting case. She can't feel envy for Carol since Alice thinks that whatever trim Carol got will only make up for what Carol lost initially by taking the trimmed piece. Alice cannot envy Donna or Bob since she goes before them.

Donna and Bob can't feel envy for anyone since they think that the four pieces are the same.

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References

[1] J. Barbanel and S. Brams. Cake division with minimal cuts: envy-free procedures for three person, four person, and beyond. *Mathematics of Social Science*, 48:251–269, 2004.

[2] S. Brams, A. Taylor, and W. Zwicker. Old and new moving knife schemes. *The Mathematical Intelligencer*, 17(4):30–35, 1995.

[3] S. Brams, A. Taylor, and W. Zwicker. A moving knife solution to the four-person envy-free cake division problem. *Proceedings of the American Mathematical Society*, 125:547–554, 1997.

[4] L. Dubins and E. Spanier. How to cut a cake fairly. *The American Mathematical Monthly*, 68:1–17, 1961.