Example of the Point of the Game

$L_7 = \{1 < 2 < 3 < 4 < 5 < 6 < 7\}$
$L_{10} = \{1 < 2 < 3 < 4 < 5 < 6 < 7 < 8 < 9 < 10\}$

1. SPOIL wants to convince DUP that $L_a \neq L_b$.
2. DUP wants to resist the attempt.

Rules of the Game

Parameter: k The number of rounds.

1. SPOIL: pick number in one orderings.
2. DUP: pick number in OTHER ORDERING. DUP will try to play a point that most 'looks like' the other point.
3. Repeat the above two steps until k round are complete.

If at the end the three points picked from L_a are in the same order as those picked from L_b then DUP wins. Otherwise SPOILER wins.

Play a student $(L_3, L_4, 2)$**,** $(L)3, L_4, 3)$
Let Students pick a, b, k **to Play**

PROBLEM 1

1. Who wins $(L_3, L_4, 2)$? (2 moves).
2. Who wins $(L_8, L_{10}, 3)$? (3 moves)
3. GENERALLY: Who wins (L_a, L_b, k).

Generalize

Can use any orderings L, L'

Play a student $\mathbb{N}$ and $\mathbb{Z}$ with 1 move, 2 moves

PROBLEM 2

In all problems we want a k such that condition holds.

1. DUP wins $(\mathrm{N}, \mathrm{Z}, k-1)$, SPOIL wins $(\mathrm{N}, \mathrm{Z}, k)$.
2. DUP wins $(\mathrm{N}, \mathrm{Q}, k-1)$, SPOIL wins $(\mathrm{N}, \mathrm{Q}, k)$.
3. DUP wins $(\mathrm{Z}, \mathrm{Q}, k-1)$, SPOIL wins $(\mathrm{Z}, \mathrm{Q}, k)$.
4. DUP wins $(L_{10}, \mathrm{N}+\mathrm{N}^*, k-1)$, SPOIL wins $(L_{10}, \mathrm{N}+\mathrm{N}^*, k)$.
5. DUP wins $(\mathrm{N}+\mathrm{Z}, \mathrm{N}, k-1)$, SPOIL wins $(\mathrm{N}+\mathrm{Z}, \mathrm{N}, k)$.

A Notion of L, L' being Similar

Let L and L' be two linear orderings.

Definition

If DUP wins the m-round DS-game on L, L' then L, L' are *m-game equivalent* (denoted $L \equiv_m^G L'$).

What is Truth?

All sentences use the usual logic symbols and $<$.

Definition

If L is a linear a linear ordering and ϕ is a sentence then $L \models \phi$ means that ϕ is true in L.

Example

Let $\phi = (\forall x)(\forall y)(\exists z)[x < y \implies x < z < y]$

1. $\mathsf{Q} \models \phi$
2. $\mathsf{N} \models \neg\phi$

Complexity of Sentences

Definition

The *quantifier depth (qd) of a sentence* is (informally) the nested depth of quantifiers.

Example

1. $(\forall x)(\forall y)(\exists z)[x < y \implies x < z < y]$ has qd 3.
2. $(\forall x)(\exists y)[y > x \wedge (\exists z)[x < y < z]$ has qd 3.

Another Notion of L, L' Similar

Let L and L' be two linear orderings.

Definition

L and L' are *m-truth-equiv* ($L \equiv_m^T L'$)

$$(\forall \phi, qd(\phi) \leq m)[L \models \phi \text{ iff } L' \models \phi.$$

The Big Theorem

Theorem

Let L, L' *be any linear ordering and let* $m \in \mathbb{N}$*. The following are equivalent.*

1. $L \equiv^{T}_{m} L'$
2. $L \equiv^{G}_{m} L'$

Applications

1. Density *cannot* be expressed with qd 2. (Proof: $Z \equiv_2^G \mathrm{Q}$ so $Z \equiv_2^T \mathrm{Q}$).
2. Well foundedness cannot be expressed in first order at all! (Proof: $(\forall n)[\mathrm{N} \, GEn \, \mathrm{N} + \mathrm{Z}]$).
3. Upshot: Questions about expressability become questions about games.