Homework 10 Morally Due April 28

FOR THIS HW IF Y IS A SET OF NUMBERS THEN $SUM(Y)$ IS THE SUM OF THE ELEMENTS IN Y. For example, if $Y = \{2, 3, 7\}$ then $SUM(Y) = 12$.

1. (0 points) Where and when is the final?

2. (30 points) Let $X = \{1, \ldots, 20\}$.

   (a) Using the pigeonhole principle show that there are 22 subsets of X of size 3, which we denote $Y_1, \ldots, Y_{22}$, such that $SUM(Y_1) = \cdots = SUM(Y_{22})$.

   (b) Using the pigeonhole principle show that there are 24 subsets of X of size 3, which we denote $Y_1, \ldots, Y_{24}$, such that $SUM(Y_1) = \cdots = SUM(Y_{24})$. (HINT: You may want to remove some subsets and remove some sums.)

   (c) You did the last two problems with the Pigeon Hole Prin, hence you did not actually FIND triples with the same sum. Find and list out 30 triples that have the same sum. (For Fun but not to hand in: See how many triples you can find that have the same sum.)

SOLUTION TO PROBLEM TWO

a) The number of subsets of X of size 3 is $\binom{20}{3} = \frac{20\times19\times18}{3\times2} = 10\times19\times6 = 1140$.

The MIN sum of three elements of X is $1 + 2 + 3 = 6$.

The MAX sum of three elements of X is $18 + 19 + 20 = 57$.

Hence the number of possible sums is $57 - 5 = 52$.

Therefore there are $\lceil \frac{1140}{52} \rceil = 22$ subsets of size 3 with the the same sum. (There could be more.)

b) Of the 1140 different subsets of size 3 there is

one that has sum 6: $\{1, 2, 3\}$.

one that has sum 7: $\{1, 2, 4\}$.

one that has sum 57: $\{18, 19, 20\}$.

one that has sum 56: $\{17, 19, 20\}$.

The number of subsets that are NOT one of those four is $1140 - 4 = 1136$.

The MIN sum of subsets that are NOT one of those four is 8.

The MAX sum of subsets that are NOT one of those four is 55.

Hence the number of possible sums is $55 - 7 = 48$.

Therefore if we restrict to all subsets of size 3 EXCEPT those four we have that there must be $\lceil \frac{1136}{48} \rceil = 24$ that have the same sum.

c) The MIN sum is 6, the MAX sum is 57, so we look for a sum thats roughly in the middle, we'll use 31. There are 42 triples that add up to 31. We list them.

$\{1, 10, 20\}$.

$\{1, 11, 19\}$.

$\{1, 12, 18\}$.

$\{1, 13, 17\}$.

$\{1, 14, 16\}$.

$\{2, 9, 20\}$.

$\{2, 10, 19\}$.

$\{2, 11, 18\}$.

$\{2, 12, 17\}$.

$\{2, 13, 16\}$.

$\{2, 14, 15\}$.

$\{3, 9, 19\}$.

$\{3, 10, 18\}$.

$\{3, 11, 17\}$.

$\{3, 12, 16\}$.

$\{3, 13, 15\}$.

$\{4, 7, 20\}$.

$\{4, 8, 19\}$.

$\{4, 9, 18\}$.

$\{4, 10, 17\}$.

$\{4, 11, 16\}$.

$\{4, 12, 15\}$.

$\{4, 13, 14\}$.

$\{5, 6, 20\}$.

$\{5, 7, 19\}$.

$\{5, 8, 18\}$.

$\{5, 9, 17\}$.

$\{5, 10, 16\}$.

$\{5, 11, 15\}$.

$\{5, 12, 14\}$.

$\{6, 7, 18\}$.

$\{6, 8, 17\}$.

$\{6, 9, 16\}$.

$\{6, 10, 15\}$.

$\{6, 11, 14\}$.

$\{6, 12, 13\}$.

$\{7, 8, 16\}$

$\{7, 9, 15\}$

$\{7, 10, 14\}$

$\{7, 11, 13\}$

$\{8, 9, 14\}$

$\{8, 10, 13\}$

$\{8, 11, 12\}$

$\{9, 10, 12\}$

END OF SOLUTION TO PROBLEM TWO

3. (30 points)

   (a) Using the pigeonhole principle show that there are 90 subsets of X, which we denote $Y_1, \ldots, Y_{90}$, such that $SUM(Y_1) = \cdots = SUM(Y_{90})$.

   (b) Find numbers a, b such that ANY 3-coloring of the $a \times b$ grid has a monochromatic rectangle.

4. (30 points)

   (a) Let $x = \{1, \ldots, 13\}$ (NOTE- I THINK I MAY HAVE ASEKD ABOUT $\{1, \ldots, 20\}$ ON THE ORIGINAL HW BUT IT SEEMS TO HAVE BEEN MANGLED WHEN I WROTE THE SOLUTION SET. I"LL DO BOTH HERE.) Using the pigeonhole principle show that there are 90 subsets of X, which we denote $Y_1, \ldots, Y_{90}$, such that $SUM(Y_1) = \cdots = SUM(Y_{90})$.

   (b) Find numbers a, b such that ANY 3-coloring of the $a \times b$ grid has a monochromatic rectangle.

SOLUTION TO PROBLEM THREE

a) IF $X = \{1, \ldots, 13\}$.

he number of subsets of X is $2^{13} = 8192$.

The MIN sum of elements of X is 0 (the empty set).

The MAX sum of elements of X is $1 + 2 + 3 + \cdots + 13 = 91$.

Hence the number of possible sums is 92.

Therefore there are $\lceil \frac{8192}{92} \rceil = 90$ subsets with the the same sum. (There could be more.)

a) IF $X = \{1, \ldots, 20\}$.

he number of subsets of X is 2^{20}.

The MIN sum of elements of X is 0 (the empty set).

The MAX sum of elements of X is $1 + 2 + 3 + \cdots + 20 = 210$.

Hence the number of possible sums is 92.

Therefore there are $\left\lceil \frac{2^{20}}{210} \right\rceil$ subsets with the the same sum.

I leave it to the student to estimate that this is at least 90 (its A LOT bigger than 90).

b) We want a column to have to have a repeat color, so we have columns of size 11. We want the number of columns to be so large that no matter how you color them you get a repeat row. So we take size $2^{11} + 1$. So the answer is $11 \times (2^{11} + 1)$.

END OF SOLUTION TO PROBLEM THREE

5. (40 points) Below we give sets of function from N to N¿ For each set say if it is FINITE or COUNTABLE or UNCOUNTABLE and PROVE it.

   (a) The set of functions f such that $(\forall x < y)[f(x) \le f(y)]$.

   (b) The set of functions f such that $(\forall x < y)[f(x) < f(y)]$.

   (c) The set of functions f such that $(\forall x < y)[f(x) \ge f(y)]$.

   (d) The set of functions f such that $(\forall x < y)[f(x) > f(y)]$.

SOLUTION TO PROBLEM FOUR

a) UNCOUNTABLE. Assume that the set is countable. Let $f_1, f_2, f_3, \ldots$ be a listing of the set. We construct a function that is IN THE SET but NOT ON THE LIST.

Its NOT GOOD ENOUGH to say $F(n) = f_n(n) + 1$ since this F might not be increasing. If you did this then you are just copying stuff you heard in class and do not really understand it. After reading and understanding this solution set you will understand it.

We define F as follows:

$F(1) = f_1(1) + 1$.

$(\forall n \ge 2)[F(n) = \max\{F(1), F(2), \ldots, F(n-1), f_n(n)\} + 1$

KEY: The way we define F it is INCREASING.

KEY: We also made sure that $(\forall n)[F(n) \neq f_n(n)]$, hence F is not on the list.

NICE EXTRA: This also solves part b.

b) UNCOUNTABLE. Use construction from part a.

c) COUNTABLE. Think about what a function in this set looks like. Here is an example:

$f(1) = 10$

$f(2) = 10$

$f(3) = 9$

$f(4) = 9$

$f(5) = 9$

$f(6) = 7$

We'll stop here. But realize that at some point the function has to be constant. The co-domain is the naturals! So the lowest you can go is 1. Hence every function in this set can be represented by a sequence of numbers that are monotone decreasing and then are constant. For example

$(10, 10, 9, 9, 9, 9, 7, 7, 7, 7, 4, 4, 4, 4, \ldots)$.

Let $CONST$ be the set of functions from N to N that are eventually constant. This is a superset of our set. We will show that $CONST$ is countable and that will show that our set is countable.

Every element of $CONST$ can be represented by a nonempty finite sequence of natural numbers which we interpret as in the following example:

$(3, 4, 9, 19)$

is the function which maps

1 to 3

2 t0 4

3 t0 9

4,5,6,..., all to 19.

So $CONST$ is the same cardinality as the set of all finite sets of naturals.

We can show that the set of all finite sets of naturals is countable by looking at

$S_1 =$ all sets of naturals that sum to 1

$S_2 =$ all sets of naturals that sum to 2

etc.

The set of all finite sets of naturals it the union of these sets. So $CONST$ is a countable union of finite sets, and hence is countable.

d) FINITE. In fact EMPTY. Try to think of an example. If a function is strictly decreasing then it must eventually get below 1.