Homework 12 REALLY Due May 12  
NOTE- MAY 12 is THE LAST DAY OF CLASS

1. (0 points) Where and when is the final?

2. (30 points) Give a clean statement of the form

\( P_I \) wins \( NIM(1, k) \) on \( n \) stones iff XXX

where XXX depends on \( k \) and \( n \).

(HINT: Work out the pattern for \( NIM(1, 2), NIM(1, 3), NIM(1, 4), NIM(1, 5), NIM(1, 6) \) until you see a pattern of patterns. Start at 0.)

SOLUTION TO PROBLEM TWO

For a NIM game let \( W(n) \) be who wins if you begin with \( n \) on the board. Recall that

\[ W(0) = II \]

For \( n \geq 1 \) \( W(n) = I \) if there exists a move getting you to a \( II \)-spot, but is \( I \) otherwise. More formally if you can remove \( \{a_1 < \cdots < a_k\} \) from the board then

\[ W(n) = II \text{ if } 0 \leq n \leq a_1 - 1. \]

For \( n \geq a_1 \)

- \( W(n) = I \) if there exists \( a_i \) such that \( n - a_i \geq 0 \) and \( W(n-a_i) = II \)
- \( W(n) = II \) otherwise.

For \( NIM(1, 2) \) here is table of who wins

<table>
<thead>
<tr>
<th>( n )</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
</tr>
</thead>
<tbody>
<tr>
<td>( W )</td>
<td>II</td>
<td>I</td>
<td>I</td>
<td>II</td>
<td>I</td>
<td>I</td>
<td>II</td>
</tr>
</tbody>
</table>

AH-HA: \( I \) wins iff \( n \equiv 1, 2 \, (\text{mod } 3) \).

For \( NIM(1, 3) \) here is table of who wins

<table>
<thead>
<tr>
<th>( n )</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
</tr>
</thead>
<tbody>
<tr>
<td>( W )</td>
<td>II</td>
<td>I</td>
<td>II</td>
<td>I</td>
<td>II</td>
<td>I</td>
<td>II</td>
</tr>
</tbody>
</table>

AH-HA: \( I \) wins iff \( n \equiv 1 \, (\text{mod } 2) \).
For NIM(1, 4) here is table of who wins

<table>
<thead>
<tr>
<th>n</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
</tr>
</thead>
<tbody>
<tr>
<td>W</td>
<td>II</td>
<td>I</td>
<td>II</td>
<td>I</td>
<td>II</td>
<td>I</td>
<td>I</td>
<td>II</td>
<td>I</td>
<td>I</td>
</tr>
</tbody>
</table>

AH-HA: I wins iff \( n \equiv 1, 3, 4 \pmod{5} \).

For NIM(1, 5) here is table of who wins

<table>
<thead>
<tr>
<th>n</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
</tr>
</thead>
<tbody>
<tr>
<td>W</td>
<td>II</td>
<td>I</td>
<td>II</td>
<td>I</td>
<td>II</td>
<td>I</td>
</tr>
</tbody>
</table>

AH-HA: I wins iff \( n \equiv 1 \pmod{2} \).

For NIM(1, 6) here is table of who wins

<table>
<thead>
<tr>
<th>n</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
<th>10</th>
<th>11</th>
<th>12</th>
<th>13</th>
</tr>
</thead>
<tbody>
<tr>
<td>W</td>
<td>II</td>
<td>I</td>
<td>II</td>
<td>I</td>
<td>II</td>
<td>I</td>
<td>I</td>
<td>II</td>
<td>I</td>
<td>I</td>
<td>II</td>
<td>I</td>
<td>I</td>
<td>I</td>
</tr>
</tbody>
</table>

AH-HA: I wins iff \( n \equiv 1, 3, 5, 6 \pmod{7} \).

Is there a pattern in all of this? YES

If \( L \equiv 0 \pmod{2} \) then

I wins iff \( n \equiv 1, 3, \ldots, L - 1 \pmod{L + 1} \) OR \( n \equiv L \pmod{L + 1} \)

If \( L \equiv 1 \pmod{2} \) then

I wins iff \( n \equiv 1 \pmod{2} \)

**END OF SOLUTION TO PROBLEM TWO**

3. (40 points) Read my notes on Mono Squares which is posted. You may use Lemma 2.4 from those notes.

(a) Show that there exists a number \( M \) such that for all 3-colorings of the \( M \times M \) grid there is a mono square.

(b) Show that for all \( c \) there exists a number \( M_c \) such that for all \( c \)-colorings of the \( M_c \times M_c \) grid there is a mono square.

**FOR SOLUTION SEE POSTED NOTES ON MONO SQUARES**
4. (30 points) Let \( A = \{1, 2, 3, 4\} \).

(a) How many relations are there over the set \( A \)?
(b) Of those, how many are functions?
(c) If I pick a relation at random what is the probability that its a function?

**SOLUTION TO PROBLEM FOUR**

A relation of \( A \) is a subset of \( A \times A \).

a) \( A \times A \) is of size 16. So there are \( 2^{16} \) relations over \( A \).

b) To count the number of functions we count:

How many numbers can 1 map to: 4
How many numbers can 2 map to: 4
How many numbers can 3 map to: 4
How many numbers can 4 map to: 4

So the answer is \( 4^4 \).

c) The probability that you pick a function is
\[
\frac{4^4}{2^{16}} = \frac{2^8}{2^{16}} = \frac{1}{2^8} \sim 0.004.
\]

JUST CURIOUS: What if the \( A = \{1, 2, 3\} \). Then the answer would be
\[
\frac{3^3}{2^6} = \frac{27}{64} \sim 0.053.
\]

JUST CURIOUS: What if the \( A = \{1, 2\} \). Then the answer would be
\[
\frac{2^2}{2^2} = \frac{1}{4} = 0.25.
\]

**END OF SOLUTION TO PROBLEM 4**