Homework Review
(FOR YOUR OWN BENEFIT HAVE IT DONE BY APRIL 7. I WILL POST SOLUTIONS THEN. NOT TO BE HANDED IN.)

1. (0 points but you have to answer) What is your name? Write it clearly. Staple your HW. When is midterm 2?

2. (20 points) Let $T(n)$ be defined as follows:

   $T(1) = 10$

   $T(2) = 15$

   $T(3) = 20$

   $(\forall n \geq 3)[T(n) = 4n + T(\lfloor n/2 \rfloor) + T(\lfloor n/4 \rfloor)]$.

   (You can ignore floor-issues in this problem and just use

   $(\forall n \geq 3)[T(n) = 4n + T(n/2) + T(n/4)]$.

   By Strong Construtive induction find a constant $a$ such that

   $(\forall n \geq 1)[T(n) \leq an]$.

   SOLUTION TO PROBLEM 2

   **Base Cases:**

   $n = 1$. $T(1) = 10$, so need $10 \leq a$, so $a \geq 10$.

   $n = 2$. $T(2) = 15$, so need $15 \leq 2a$, so $a \geq 7.5$

   $n = 3$. $T(3) = 20$, so need $20 \leq 3a$, so $a \geq 6.6$

   HENCE $a \geq 10$.

   **IH:** We can assume $(\forall n' < n)[T(n') \leq an']$.

   **IS:**

   $$T(n) = 4n+T(\lfloor n/2 \rfloor)+T(\lfloor n/4 \rfloor)] \leq 4n+T(n/2)+T(n/4) \leq 4n+an/2+an/4$$

   We NEED

   $4n + an/2 + an/4 \leq an$

   $4 + a/2 + a/4 \leq a$

   $4 \leq 3a/4$

$16 \leq 3a$

$a \geq 16/3 \sim 5.3.$

SO the two conditions are $a \geq 10$ and $a \geq 5.3$. We can take $a = 10$.

END OF SOLUTION TO PROBLEM 2

3. (20 points) Give a Context Free Grammar for the language

$$\{a^n b^{2n} : n \in \mathsf{N}\}$$

4. (20 points) Give a 2-coloring of the edges of $K_4$ such that there are no monochromatic triangles.

5. (20 points) In class and on the slides there are FOUR proofs that:

$$(\forall n \geq 3)(\exists d_1 < \cdots < d_n)\left[1 = \frac{1}{d_1} + \cdots + \frac{1}{d_n}\right].$$

   Each proof leads to a an actual way to, for all $n$, express 1 as the sum of $n$ distinct reciprocals. Guided by these proofs find four ways to write 1 as the sum of 5 reciprocals.

   SOLUTION TO PROBLEM 5

   All four proofs start with:

   $1 = \frac{1}{2} + \frac{1}{3} + \frac{1}{6}$ (the $d = 3$ case)

   PROOF 1 uses

   $\frac{1}{d} = \frac{1}{d+1} + \frac{1}{d(d+1)}$ (use twice).

   $1 = \frac{1}{2} + \frac{1}{3} + \frac{1}{6}$

   Use $\frac{1}{6} = \frac{1}{7} + \frac{1}{42}$ to get:

   $= \frac{1}{2} + \frac{1}{3} + \frac{1}{7} + \frac{1}{42}$

   Use $\frac{1}{42} = \frac{1}{43} + \frac{1}{42 \times 43}$ to get:

   $= \frac{1}{2} + \frac{1}{3} + \frac{1}{7} + \frac{1}{43} + \frac{1}{1806}$

   PROOF 2 uses

   $\frac{1}{d} = \frac{1}{2d} + \frac{1}{3d} + \frac{1}{6d}$ (we once)

$1 = \frac{1}{2} + \frac{1}{3} + \frac{1}{6}$

Use $\frac{1}{6} = \frac{1}{12} + \frac{1}{18} + \frac{1}{36}$

To get

$1 = \frac{1}{2} + \frac{1}{3} + \frac{1}{12} + \frac{1}{18} + \frac{1}{36}$

PROOF 3 uses

$\frac{1}{d} = \frac{1}{3d/2} + \frac{1}{3d}$ (use twice).

$1 = \frac{1}{2} + \frac{1}{3} + \frac{1}{6}$

Use $\frac{1}{6} = \frac{1}{9} + \frac{1}{18}$ to get

$1 = \frac{1}{2} + \frac{1}{3} + +\frac{1}{9} + \frac{1}{18}$

Use $\frac{1}{18} = \frac{1}{27} + \frac{1}{54}$ to get

$1 = \frac{1}{2} + \frac{1}{3} + \frac{1}{9} + \frac{1}{27} + \frac{1}{54}$

PROOF 4 uses

$1 = \frac{1}{2} + \frac{1}{3} + \frac{1}{6}\left(\frac{1}{d_1} + \cdots + \frac{1}{d_n}\right)$. (use Once)

$1 = \frac{1}{2} + \frac{1}{3} + \frac{1}{6}$

So

$1 = \frac{1}{2} + \frac{1}{3} + \frac{1}{6}\left(\frac{1}{2} + \frac{1}{3} + \frac{1}{6}\right)$.

$=$

$\frac{1}{2} + \frac{1}{3} + \frac{1}{12} + \frac{1}{18} + \frac{1}{36}$

END OF SOLUTION TO PROBLEM 5