On Using and Not Using the Pigeon Hole Principle

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1 Introduction

Consider the following problem: Show that x elements of $\binom{[n]}{3}$ have the same sum, try to make x as large as possible. We do this with the PHP and without and see what happens. The upshot is that the PHP proofs are short, but without it we can get exact results which are better.

2 Using PHP

The quick way to do this is via the PHP. The number of sets is $\binom{n}{3}$. The least sum is 3, the largest sum is $3n-3$. Hence there must be

$$\left\lceil \frac{\binom{n}{3}}{3n-3} \right\rceil = \left\lceil \frac{n(n-1)(n-2}{6(3n-3)} \right\rceil = \left\lceil \frac{n(n-2)}{18} \right\rceil = \left\lceil \frac{n^2}{18} - \frac{n}{9} \right\rceil$$

For $n = 99$ this yields 534.

Can we do better? Yes- we know that the small and large sums are not going to occur much, so we can remove the sets that make them occur, and also remove those sums.

Removing the sums $1+2+3=6$ and $n-2+n-1+n=3n-3$, and the sets $\{1,2,3\}$ and $\{n-2,n-1,n\}$.

$$\left\lceil \frac{\binom{n}{3}-2}{(3n-3)-2} \right\rceil = \left\lceil \frac{\frac{n(n-1)(n-2)}{6}-2}{3n-5} \right\rceil = \left\lceil \frac{\frac{n(n-1)(n-2)-12}{6}}{3n-5} \right\rceil = \left\lceil \frac{n^3-3n^2+2n-12}{18n-30} \right\rceil$$

For $n = 99$ this yields 538.

Removing the sums $1+2+3 = 6$, $1+2+4 = 7$, $n-2+n-1+n = 3n-3$, $n-3+n-1+n = 3n-4$ and the four sets they corresond to yields

$$\left\lceil \frac{\binom{n}{3} - 4}{(3n-3)-4} \right\rceil = \left\lceil \frac{\frac{n(n-1)(n-2)}{6} - 4}{3n-7} \right\rceil = \left\lceil \frac{\frac{n(n-1)(n-2)-24}{6}}{3n-7} \right\rceil = \left\lceil \frac{n^3 - 3n^2 + 2n - 24}{18n - 42} \right\rceil$$

For $n = 99$ this yields 540

Removing the sums $1+2+3 = 6$, $1+2+4 = 7$, $1+2+5 = 1+3+4 = 8$, and the analogous three sums on the upper end, so thats 6 sums and 8 sets yields

$$\left\lceil \frac{\binom{n}{3} - 8}{(3n-3)-6} \right\rceil = \left\lceil \frac{\frac{n(n-1)(n-2)}{6} - 8}{3n-9} \right\rceil = \left\lceil \frac{\frac{n(n-1)(n-2)-48}{6}}{3n-9} \right\rceil = \left\lceil \frac{n^3 - 3n^2 + 2n - 48}{18n - 54} \right\rceil$$

For $n = 99$ this yields 544.

Removing the sums $1+2+3 = 6$, $1+2+4 = 7$, $1+2+5 = 1+3+4 = 8$, $1+2+6 = 1+3+5 = 2+3+4 = 9$. and the sets, and the ones on the high end, so thats 8 sums and 12 sets.

$$\left\lceil \frac{\binom{n}{3} - 12}{(3n-3)-8} \right\rceil = \left\lceil \frac{\frac{n(n-1)(n-2)}{6} - 12}{3n-11} \right\rceil = \left\lceil \frac{\frac{n(n-1)(n-2)-72}{6}}{3n-11} \right\rceil = \left\lceil \frac{n^3 - 3n^2 + 2n - 72}{18n - 66} \right\rceil$$

For $n = 99$ this yields 549

If I keep doing this I'll keep getting formulas that are a bit better but still asymptotically $\frac{n^2}{18}$. Is this the real answer asymptotially? No.

3 Not using PHP

We can actually get an exact count, though we will assume $n \equiv 4 \pmod 8$ which will simplify some of the calculations.

Note that the smallest sum is 6 and the largest sum is 3n-3. Hence the most common sum should be at about the midpoint. We find out how many elements of $\binom{[n]}{3}$ that sum to $\frac{3n}{2}$.

We write all elements of $\binom{[n]}{3}$ as a three-tuple (x, y, z) where $x < y < z$. There are two kinds of elements of $\binom{[n]}{3}$ that sum to $\frac{3n}{2}$. The first type is of the form $(i, \frac{n}{2} - i + x, n - x)$ where $1 \leq i \leq \frac{n}{4} - 1$ and $0 \leq x \leq \frac{n-2i-3+(-1)^i}{4}$. Note that the conditions on i, x ensures that $i < \frac{n}{2} - i + x < n - x$.

How many such triples are there? Let $n = 8L + 4$.

$$\frac{1}{4} \sum_{i=1}^{(n-4)/4} n - 2i - 3 + (-1)^i = \frac{1}{4} \left(\sum_{i=1}^{(n-4)/4} (n-3) - 2 \sum_{i=1}^{(n-4)/4} i + \sum_{i=1}^{(n-4)/4} (-1)^i \right)$$

$$= \frac{1}{4} \left(\frac{(n-4)(n-3)}{4} - \frac{n-4}{4} \frac{n}{4} \right)$$