What is a proof?

A good proof should have:

- A clear statement of what is to be proved (labeled as Theorem, Lemma, Proposition, or Corollary).
- The word "Proof" to indicate where the proof starts.
- A clear indication of flow.
- A clear justification for each step.
- A clear indication of the conclusion.
- The abbreviation "QED" ("Quod Erat Demonstradum" or "that which was to be proved") or equivalent to indicate the end of the proof.

Summary of Proof Methods

- Direct proof
- Proof by contraposition
- Proof by contradiction
- Exhaustive Proof
- Proof by cases

Statement of Theorems

The following are equivalent:

- The sum of two positive integers is positive.
- If m, n are positive integers then their sum $m + n$ is a positive integer.
- For all positive integers m, n their sum $m + n$ is a positive integer.
- $(\forall m, n \in \mathbb{Z})[((m > 0) \wedge (n > 0)) \rightarrow ((m + n) > 0)]$

Number Definitions

Definition

An integer n is *even* if $n = 2k$ for some integer k, and is *odd* if $n = 2k + 1$ for some integer k.

Definition

A number q is *rational* if there exist integers a, b with $b \neq 0$ such that $q = a/b$.

Definition

A real number that is not rational is *irrational*.

Closure

- $\mathbb{Z}$ is closed under addition.

  If $a, b \in \mathbb{Z}$ then $a + b \in \mathbb{Z}$.

- $\mathbb{Q}^{\neq 0}$ is closed under division.

  If $q, r \in \mathbb{Q}^{\neq 0}$ then $\frac{q}{r} \in \mathbb{Q}^{\neq 0}$.

- $\mathbb{Z}^{\neq 0}$ is *not* closed under division.

  $\frac{3}{5} \notin \mathbb{Z}^{\neq 0}$.

Direct Proofs

- The square of an even number is even.
- The product of two odd numbers is odd.
- The sum of two rational numbers is rational.

Do in class.

Proof by Contraposition

- If $3n + 2$ is odd, where n is an integer, then n is odd.
- If n^2 is even, where n is an integer, then n is even.
- If $n = ab$, where a and b are positive integers, then $a \leq \sqrt{n}$ or $b \leq \sqrt{n}$. What is the domain for n?

Do in class.

Proof by Contradiction

- At least four of any 22 days must fall on the same day of the week.
- $\sqrt{2}$ is irrational.

Do in class.

Proofs of Equivalence

- If n is an integer, then n is odd if and only if n^2 is odd.
- The following statements about the integer n are equivalent:
  - n is even.
  - $n - 1$ is odd.
  - n^2 is even.

Do in class.

Exhaustive Proofs

- For all positive integers n with $n \leq 4$, $(n+1)^3 \geq 3^n$.
- There are no integer solutions to the equation $x^2 + 3y^2 = 8$.

Do in class.

Interesting vs Boring: Round I

- Every number ≤ 100 can be written as the sum of nine cubes. Can be proven by exhaustion. **boring proof of boring statement**.
- Every number can be written as the sum of nine cubes. Can be proven by **very interesting mathematics**
- Is nine optimal? Yes: 23 requires nine cubes.

Do in Class

Interesting vs Boring: Round II

- Every number can be written as the sum of nine cubes. Nine is optimal since 23 requires nine.
- If every number except 23 can be written as the sum of eight cubes that seems **more interesting.**

What is the Right Question to Ask?

Proofs by Cases

- For every integer n, $n^2 \geq n$.
- If n is odd then $n^2 = 8m + 1$ for some integer m.

Do in class.

Existence Proofs

There exists a postive integer that is the sum of two cubes of postive integers in two different ways.

$$(\exists p, q, r, s, n \in \mathbb{Z}^+)$$
$$[(p \neq q) \wedge (p \neq r) \wedge (p \neq s)$$
$$\wedge (q \neq r) \wedge (q \neq s) \wedge (r \neq s)$$
$$\wedge (n = p^3 + q^3) \wedge (n = r^3 + s^3)]$$

Tell Story

Existence Proofs

$$(\exists x, y \in \mathbb{R} - \mathbb{Q})[x^y \in \mathbb{Q}]$$

Do in class.