What is Logic?

Definition

Logic is the study of valid reasoning.

- Philosophy
- Mathematics
- Computer science

Definition

Mathematical Logic is the mathematical study of the methods, structure, and validity of mathematical deduction and proof. [Wolfram Mathworld]

Propositions

Definition

A *proposition* is a declaritive sentence that is either true or false.

- Today is Tuesday.
- $5 + 2 = 7$
- $3 \cdot 6 > 18$
- Why must I be a teenager in love?
- Triangles.
- Two members of the faculty have the same birthday.
- The current king of France is bald.

AND

Definition

The *AND* of two propostions, p and q, is the proposition "p and q". It is true when both p and q are true. We denote it $p \wedge q$.

Example

s: The sky is blue.

g: The grass is green.

m: The moon is made of cheese.

$s \wedge g$: The sky is blue and the grass is green.

$s \wedge m$: The sky is blue and the moon is made of cheese.

OR

Definition

The *OR* of two propostions, p and q, is the proposition "p or q". It is true when either p or q is true. We denote it $p \vee q$.

Example

s: The sky is blue.

g: The grass is red.

m: The moon is made of cheese.

$s \vee g$: The sky is blue or the grass is red.

$g \vee m$: The grass is red or the moon is made of cheese.

NOT

Definition

The *NOT* of one propostions, p is the proposition "NOT p" It is true when p is false. We denote it $\neg p$.

Example

s: The sky is blue.

g: The grass is red.

$\neg s$: The sky is not blue.

$\neg g$: The grass is not red.

Propositional Formulas (Prop Fml)

Definition

Informal A *Prop Fml* is variables strung together with $\wedge$'s, $\vee$'s, and $\neg$'s.

Formal

1) Any variable is a *Prop Fml*

2) If $\phi_1(\vec{x})$ and $\phi_2(\vec{x})$ are Prop Fml then the forllowing are *Prop Fml*:

$(\phi_1(\vec{x}) \wedge \phi_2(\vec{y}))$,

$(\phi_1(\vec{x}) \vee \phi_2(\vec{y}))$,

$\neg\phi_1(\vec{x})$

Truth tables (TT)

The meaning of a logical operation can be expressed as its “truth table.”

- Construct the TT for AND.
- Construct the TT for OR.
- Construct the TT for NOT.

Do in class.

A worked example

$$(\neg s \wedge t) \vee \neg t.$$

Do TT in class.

XOR

The word "or" is often used to mean "one or the other," but this is *not* the same meaning of "or" in logic!

Definition

The *XOR* of two statements p and q is true when either p is true or q is true, but not both. We denote it $p \wedge q$.

p	q	$p \oplus q$
T	T	F
T	F	T
F	T	T
F	F	F

Tell Vegetarian Story-Trader Joe's

Logical equivalences

How do we know if two logical statements are equivalent?

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- Construct truth tables for each.
- Check if final columns match.

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Theorem

Let p and q be statement variables. Then

$$\begin{aligned} & (p \vee q) \wedge \neg(p \wedge q) \equiv p \oplus q \\ \textit{and} \quad & (p \wedge \neg q) \vee (q \wedge \neg p) \equiv p \oplus q \, . \end{aligned}$$

Prove in class (using Truth Tables).

Conditional Statements

Hypothesis $\rightarrow$ Conclusion

Example

- If it is raining, I will carry my umbrella.
- If you don't eat your dinner, you will not get dessert.

p	q	$p \rightarrow q$
T	T	T
T	F	F
F	T	T
F	F	T

Expressing Conditionals

Conditional can be expressed in many ways:

- if p then q
- p implies q
- q if p
- p only if q
- a sufficient condition for q is p
- a necessary condition for p is q

More on Conditional

In logic the hypothesis and conclusion need not relate to each other.

Example

- If Joe likes cats, then the sky is blue.
- If Joe likes cats, then the moon is made of cheese.

In programming languages "if-then" is a command.

Example

- If it rains today, then buy an umbrella.
- If $x > y$ then $z := x + y$

Contrapositive

Definition

The *contrapositive* of a conditional statement is obtained by transposing its conclusion with its premise and inverting. So,

Contrapositive of $p \to q$ is $\neg q \to \neg p$.

Example

Original statement: *If I live in College Park, then I live in Maryland.*

Contrapositive: *If I don't live in Maryland, then I don't live in College Park.*

Theorem

The contrapositive of an implication is equivalent to the original statement.

Prove in class (using Truth Tables)

Prove in class (using Tortured English)

Converse

Definition

The *converse* of a conditional statement is obtained by transposing its conclusion with its premise.

Converse of $p \to q$ is $q \to p$.

Example

Original statement: *If I live in College Park, then I live in Maryland.*

Converse: *If I live in Maryland, then I live in College Park.*

The converse is **NOT** equiv to the original.

Prove in class (using Truth Tables).

Prove in class (using common sense).

Biconditional Statements

Example

- I will carry my umbrella, if and only if it is raining.
- You will get dessert, if and only if you eat your dinner.

p	q	$p \leftrightarrow q$
T	T	T
T	F	F
F	T	F
F	F	T

An Applications of Prop Logic

In many programming languages there are conditionals.

Example

If $(x \le y + 3$ AND $z \ge 0)$ OR $y \le z$ then $\cdots$

Logic useful to figure out what you want.

Can Simplify

If sentences of the form $t_1 \leq t_2$ or $t_1 < t_2$ then negations are easy and AND's may become easy:

1. $\neg(t_1 \leq t_2) \equiv t_1 > t_2$
2. $\neg(t_1 < t_2) \equiv t_1 \geq t_2$
3. $t_1 < t_2 \wedge t_2 \leq t_3 \equiv t_1 < t_2 \leq t_3$

Laws of Logic

Given any statement variables p, q, and r, a tautology t and a contradiction c, the following logical equivalences hold:		
1. Commutative laws:	$p \wedge q \equiv q \wedge p$	$p \vee q \equiv q \vee p$
2. Associative laws:	$(p \wedge q) \wedge r \equiv p \wedge (q \wedge r)$	$(p \vee q) \vee r \equiv p \vee (q \vee r)$
3. Distributive laws:	$p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$	$p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$
4. Identity laws:	$p \wedge t \equiv p$	$p \vee c \equiv p$
5. Negation laws:	$p \vee \neg p \equiv t$	$p \wedge \neg p \equiv c$
6. Double Negative law:	$\neg(\neg p) \equiv p$	
7. Idempotent laws:	$p \wedge p \equiv p$	$p \vee p \equiv p$
8. DeMorgan's laws:	$\neg(p \wedge q) \equiv \neg p \vee \neg q$	$\neg(p \vee q) \equiv \neg p \wedge \neg q$
9. Universal bounds laws:	$p \vee t \equiv t$	$p \wedge c \equiv c$
10. Absorption laws:	$p \vee (p \wedge q) \equiv p$	$p \wedge (p \vee q) \equiv p$
11. Negations of t and c:	$\neg t \equiv c$	$\neg c \equiv t$

Use Laws to Simplify

Use Laws to Simplify Fmls. Especially De Morgans and Dist.

$$\neg(\neg x \wedge y) \vee (x \vee \neg y)$$

Simplify

The SAT Problem

Given a prop fml $\phi(x_1, \ldots, x_n)$ is there some way to assign T and F to the variables so that it comes out true.

Discuss How to Solve this Problem