1. (0 points but please DO IT) What is your name?

2. (20 points- AND if you got the Dup-Spoiler Question wrong on the untimed midterm, but get THIS question right, you will get FULL CREDIT on the that question)

For this problem you may ASSUME the following

- For all \( n \), for all \( a \geq 2^n \), DUP wins \((\mathbb{N} + \mathbb{N^*}, L_a; n)\).
- For all \( n \), for all \( a \geq 2^n \), DUP wins \((\mathbb{N} + \mathbb{Z} + \mathbb{N^*}, L_a; n)\).
- For all \( n \), for all \( a \geq 2^n \), DUP wins \((\mathbb{N} + \mathbb{Z} + \mathbb{Z} + \mathbb{N^*}, L_a; n)\).
- For all \( n \), DUP wins \((\mathbb{N}, \mathbb{N} + \mathbb{Z}; n)\).

And NOW the question. Prove the following rigorously, similar to the end of the slides on DUP SPOILER games.

(a) For all \( n \), DUP wins the \((\mathbb{N}, \mathbb{N} + \mathbb{Z} + \mathbb{Z}; n)\) game.

(b) For all \( n \), DUP wins the \((\mathbb{N}, \mathbb{N} + \mathbb{Z} + \mathbb{Z} + \mathbb{Z}; n)\) game. (You can use Part a)
3. (40 points) Let $a_n$ be defined by

$a_1 = 10$

$a_2 = 20$

$a_3 = 30$

$(\forall n \geq 4)[a_n = a_{n-1} + 2a_{n-2} + 3a_{n-3}]$.

Using constructive induction find NATURAL NUMBERS $A, B$ such that

$(\forall n \geq 1)[a_n \leq AB^n]$.
4. (40 points) In this problem $\frac{n}{2}$ means $\left\lfloor \frac{n}{2} \right\rfloor$. In this problem we will be looking at the recurrence

$$a_1 = 1$$

$$(\forall n \geq 2)[a_n = a_{n-1} + a_{n/2}].$$

(a) (0 points but you will need it for the later parts) Write a program that does the following:

On input $d, N$ determine

For how many $1 \leq n \leq N$ is $a_n \equiv 0 \pmod{d}$.

For how many $1 \leq n \leq N$ is $a_n \equiv 1 \pmod{d}$.

For how many $1 \leq n \leq N$ is $a_n \equiv 2 \pmod{d}$.

: 

For how many $1 \leq n \leq N$ is $a_n \equiv d - 1 \pmod{d}$.

(Advice: Compute $a_n \pmod{d}$ instead of $a_n$ to avoid large numbers.)
(b) (20 points) Run your program for \(N = 1000\) and \(d = 2, 3, \ldots, 20\). Present your data as follows (the numbers below are made up)

\[d = 2\]

\[
\begin{array}{l|l}
   c & \{\{n: n \equiv c \pmod{2}\}\}\ |
\end{array}
\]

\[
\begin{array}{c|c}
   0 & 410 \\
   1 & 590 \\
\end{array}
\]

\[d = 3\]

\[
\begin{array}{l|l}
   c & \{\{n: n \equiv c \pmod{3}\}\}\ |
\end{array}
\]

\[
\begin{array}{c|c}
   0 & 333 \\
   1 & 333 \\
   2 & 334 \\
\end{array}
\]

\[
\vdots \quad \vdots \quad \vdots 
\]

\[d = 20\]

\[
\begin{array}{l|l}
   c & \{\{n: n \equiv c \pmod{20}\}\}\ |
\end{array}
\]

\[
\begin{array}{c|c}
   0 & 100 \\
   1 & 0 \\
   2 & 100 \\
   3 & 0 \\
   4 & 25 \\
   5 & 25 \\
   6 & 25 \\
   7 & 25 \\
   8 & 100 \\
   9 & 0 \\
  10 & 100 \\
  11 & 0 \\
  12 & 25 \\
  13 & 25 \\
  14 & 25 \\
  15 & 25 \\
  16 & 100 \\
  17 & 100 \\
  18 & 200 \\
  19 & 0 \\
\end{array}
\]

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(c) (20 points) Based on your data make a conjecture of the form:

Let $c, d$ be such that $0 \leq c \leq d - 1$ and $d \geq 2$. There exists an infinite number of $n$ such that $a_n \equiv c \pmod{d}$ IFF XXX($c, d$).