START

RECORDING
Intro to Combinatorics
(“that n choose 2 stuff”)

CMSC 250
Jason’s sandwich
Jason’s Sandwich

Suppose that Jason has the following ingredients to make a sandwich with:
- White or black bread
- Butter, Mayo or Honey Mustard
- Romaine Lettuce, Spinach, Kale
- Bologna, Ham or Turkey
- Tomato or egg slices
Jason’s Sandwich

• Suppose that Jason has the following ingredients to make a sandwich with:
  • White or black bread
  • Butter, Mayo or Honey Mustard
  • Romaine Lettuce, Spinach, Kale
  • Bologna, Ham or Turkey
  • Tomato or egg slices
• How many different sandwiches can Jason make?
Jason’s Sandwich

- Suppose that Jason has the following ingredients to make a sandwich with:
  - White or black bread  2 options
  - Butter, Mayo or Honey Mustard  3 options
  - Romaine Lettuce, Spinach, Kale  3 options
  - Bologna, Ham or Turkey  3 options
  - Tomato or egg slices  2 options
- **How many different sandwiches can Jason make?**
  - \[ 2 \times 3 \times 3 \times 3 \times 2 = 4 \times 27 = 108 \]
The Multiplication Rule

• Suppose that $E$ is some experiment that is conducted through $k$ sequential steps $s_1, s_2, \ldots, s_k$, where every $s_i$ can be conducted in $n_i$ different ways.
The Multiplication Rule

• Suppose that $E$ is some experiment that is conducted through $k$ sequential steps $s_1, s_2, ..., s_k$, where every $s_i$ can be conducted in $n_i$ different ways.
  • Example: $E =$ “sandwich preparation”, $s_1 =$ “chop bread”, $s_2 =$ “choose condiment”, ...
The Multiplication Rule

• Suppose that $E$ is some experiment that is conducted through $k$ sequential steps $s_1, s_2, \ldots, s_k$, where every $s_i$ can be conducted in $n_i$ different ways.
  
  • Example: $E = \text{“sandwich preparation”}$, $s_1 = \text{“chop bread”}$, $s_2 = \text{“choose condiment”}$, ...

• Then, the total number of ways that $E$ can be conducted in is

\[
\prod_{i=1}^{k} n_i = n_1 \times n_2 \times \cdots \times n_k
\]
A Familiar Example

• How many subsets are there of a set of 4 elements?
• Example: \{a, b, c, d\}
  • a: in or out. 2 choices.
  • b: in or out. 2 choices.
  • c: in or out. 2 choices.
  • d: in or out. 2 choices.
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\[2 \times 2 \times 2 \times 2 = 2^4 = 16\] subsets.
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  • d: in or out. 2 choices.
  \[2 \times 2 \times 2 \times 2 = 2^4 = 16\]
  subsets.

• Generalization: there are \(2^n\) subsets of a set of size \(n\).
  • But you already knew this.
Permutations

• Consider the string “machinery”.
Permutations

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• A permutation of “machinery” is a string which results by re-organizing the characters of “machinery” around.
Permutations

• Consider the string “machinery”.
• A permutation of “machinery” is a string which results by re-organizing the characters of “machinery” around.
  • Examples: chyirenma, hcyranemi, machinery (!)
  • Question: How many permutations of “machinery” are there?
# Permutations

m a c h i n e r y

9 options for 'm'
# Permutations

machinery

9 options for ‘m’
# Permutations

machinery

8 options for ‘a’
# Permutations

8 options for ‘a’
# Permutations

machinery

_ _ __ _ _ a _ _

7 options for ‘c’...
# Permutations

machinery

_ _ m _ _ c a _ _

7 options for ‘c’...
# Permutations

machinery

_ _ m _ _ c _ _

6 options for ‘h’…
# Permutations

machinery

6 options for ‘h’...

h _ m _ _ c a _ _
# Permutations

machinery

h_ _m_ _c a_ _

5 options for ‘i’
# Permutations

machinery

h  m  c a  i

5 options for ‘i’
# Permutations

machinery

h __ m __ c a __ i

4 options for ‘n’
# Permutations

machinery

4 options for ‘n’

h__m__n__c__a__i
# Permutations

machinery

h_ m_ nca_ i

3 options for ‘e’
# Permutations

machinery

3 options for ‘e’

hemnca
# Permutations

machinery

2 options for ‘r’

hemncai
# Permutations

machinery

2 options for ‘r’

h e m _ n c a r i
# Permutations

machinery

hemncarij

1 option for ‘y’
# Permutations

machinery

1 option for ‘y’
# Permutations

Total possible permutations = \( 9 \times 8 \times \cdots \times 2 \times 1 = 9! = 362880 \)
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That’s a lot! (Original string has length 9)
# Permutations

Total #possible permutations \(= 9 \times 8 \times \cdots \times 2 \times 1 = 9! = 362880\)

In general, for a string of length \(n\) we have ourselves \(n!\) different permutations!

That’s a lot! (Original string has length 9)
Permutations

• Now, consider the string “puzzle”.
Permutations

• Now, consider the string “puzzle”.
• How many permutations are there of this string?
Permutations

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• How many permutations are there of this string?
• Note that two letters in puzzle are the same.
Permutations

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• Note that two letters in puzzle are the same.
  • Call the first $z_{1}$ and the second $z_{2}$.
Permutations

• Now, consider the string “puzzle”.
• How many permutations are there of this string?
• Note that two letters in puzzle are the same.
  • Call the first $z_{1}$ and the second $z_{2}$
• So, one permutation of $puz_{1}z_{2}le$ is $puz_{2}z_{1}le$
Permutations

• Now, consider the string “puzzle”.
• How many permutations are there of this string?
• Note that two letters in puzzle are the same.
  • Call the first $z_1$ and the second $z_2$
• So, one permutation of $puz_1z_2le$ is $puz_2z_1le$
  • But this is clearly equivalent to $puz_1z_2le$, so we wouldn’t want to count it!
Permutations

• Now, consider the string “puzzle”.
• How many permutations are there of this string?
• Note that two letters in puzzle are the same.
  • Call the first z $z_1$ and the second z $z_2$
• So, one permutation of $puz_1z_2le$ is $puz_2z_1le$
  • But this is clearly equivalent to $puz_1z_2le$, so we wouldn’t want to count it!
  • So clearly the answer is not 6! (6 is the length of “puzzle”)
Permutations

• Now, consider the string “puzzle”.
• How many permutations are there of this string?
• Note that two letters in puzzle are the same.
  • Call the first \( z_1 \) and the second \( z_2 \)
• So, one permutation of \( pu\bar{z}_1z_2le \) is \( pu\bar{z}_2z_1le \)
  • But this is clearly equivalent to \( pu\bar{z}_1z_2le \), so we wouldn’t want to count it!
  • So clearly the answer is not 6! (6 is the length of “puzzle”)  
  • What is the answer?
Thought Experiment

• Pretend the two ‘z’s in “puzzle” are different, e.g $z_1, z_2$
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  • Then, 6! permutations, as discussed
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  • Now we have the “equivalent” permutations, for instance

$z_1 pu l z_2 e$
$z_2 pu l z_1 e$
Thought Experiment

• Pretend the two ‘z’s in “puzzle” are different, e.g $z_1, z_2$
  • Then, 6! permutations, as discussed
  • Now we have the “equivalent” permutations, for instance

\[
\begin{align*}
z_1pulz_2e \\
z_2pulz_1e
\end{align*}
\]

• We want to not doublecount these!
Thought Experiment

\[ z_1 p u l z_2 e \]
\[ z_2 p u l z_1 e \]

We want to **not doublecount** such permutations!

• Then, we need to stop pretending that the ‘z’s are **different**
Thought Experiment

\[ z_1 pulz_2 e \]
\[ z_2 pulz_1 e \]

We want to **not doublecount** such permutations!

- Then, we need to stop pretending that the ‘z’ s are **different**
  - **Bad news: 6! is overcount 😞**
Thought Experiment

\[ z_1 pulz_2 e \]
\[ z_2 pulz_1 e \]

We want to \textbf{not doublecount} such permutations!

• Then, we need to stop pretending that the ‘z’s are \textbf{different}
  • \textbf{Bad news: 6! is overcount 😞}
  • \textbf{Good news: 6! is an overcount in a precise way! 😊 Everything is counted exactly twice!}
Thought Experiment

\[ z_1pulz_2e \]
\[ z_2pulz_1e \]

We want to **not doublecount** such permutations!

- Then, we need to stop pretending that the ‘z’s are different
  - **Bad news:** 6! is overcount 😞
  - **Good news:** 6! is an overcount in a precise way! 😊 Everything is counted **exactly twice**!
- **Answer:** \( \frac{6!}{2} \)
Permutations

• Now, consider the string “scissor”.
Permutations

• Now, consider the string “scissor”.
• How many permutations of “scissor” are there?
Permutations

• Now, consider the string “scissor”.
• How many permutations of “scissor” are there?
• Note that three letters in “scissor” are the same.
  • As previously discussed, the answer cannot be $7!$ (7 is the length of “scissor”)
Permutations

• Now, consider the string “scissor”.
• How many permutations of “scissor” are there?
• Note that three letters in “scissor” are the same.
  • As previously discussed, the answer cannot be $7!$ (7 is the length of “scissor”)
  • Observe all the possible positions of the various ‘s’s:
    • $s_1cis_2s_3or$
    • $s_1cis_3s_2or$
    • $s_2cis_1s_3or$
    • $s_2cis_3s_1or$
    • $s_3cis_1s_2or$
    • $s_3cis_2s_1or$
Permutations

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    • $s_3cis_1s_2or$
    • $s_3cis_2s_1or$

$3! = 6$ different ways to arrange those $3 \text{ ‘s’s}$
Final Answer

• Think of it like this: How many times can I fit essentially the same string into the number of permutations of the original string?
Final Answer

• Think of it like this: *How many times can I fit essentially the same string into the number of permutations of the original string?*

• Therefore, the total #permutations when not assume different ‘s’s is

\[
\frac{7!}{3!} = \frac{1 \times 2 \times 3 \times 4 \times 5 \times 6 \times 7}{1 \times 2 \times 3} = 20 \times 42 = 840
\]
Complex Overcounting

• Consider now the string “onomatopoeia”.
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• 12 letters, with 4 ‘o’s, 2 ‘a’s
Complex Overcounting

• Consider now the string “onomatopoeia”.
• 12 letters, with 4 ‘o’s, 2 ‘a’s
• Considering the characters being different, we have:

\[ o_1n_o_2m_a_0_3p_o_4e_i_0, \]
Complex Overcounting

• Consider now the string “onomatopoeia”.
• 12 letters, with 4 ‘o’s, 2 ‘a’s
• Considering the characters being different, we have:

\[ o_1 n o_2 m a t o_3 p o_4 e i a, \]
\[ o_1 n o_2 m a t o_4 p o_3 e i a, \]
Complex Overcounting

• Consider now the string “onomatopoeia”.
• 12 letters, with 4 ‘o’s, 2 ‘a’s
• Considering the characters being different, we have:

$$o_1n_2m_0t_4p_0_4e_ia,$$
$$o_1n_2m_0t_4p_0_3e_ia,$$
$$o_3m_0_4p_0_3e_ia,$$
$$o_1n_3m_0_4p_0_2e_ia,$$
• Consider now the string “onomatopoeia”.
• 12 letters, with 4 ‘o’s, 2 ‘a’s
• Considering the characters being different, we have:

How many such positionings of the ‘o’s are possible?

6 12 16 Something Else

\[ o_1 n_2 m_0 a_3 p_0 4 e_1 a, \]
\[ o_1 n_0 m_2 a_0 4 p_0 3 e_1 a, \]
\[ o_1 n_0 m_3 a_0 4 p_0 2 e_1 a, \]
\[ \ldots \]
Complex Overcounting

• Consider now the string “onomatopoeia”.
• 12 letters, with 4 ‘o’s, 2 ‘a’s
• Considering the characters being different, we have:

How many such positionings of the ‘o’s are possible?

\[ p_1 n n p_2 c c s s p_3 p_4 p_5 p_6 \]

\[ p_1 n n p_3 c c s s p_4 p_5 p_6 \]

\[ p_1 n n p_5 c c s s p_3 p_4 p_6 \]

\[ … \]

\[ 4! = 24 \text{ different ways.} \]
Complex Overcounting

• However, we also have the two ‘a’s to consider!
Complex Overcounting

• However, we also have the two ‘a’ s to consider!
• Fortunately, those equivalent permutations are simpler to count:

\[ \text{onom} a_1 \text{topoeia} a_2 \]
\[ \text{onom} a_2 \text{topoeia} a_1 \]
Complex Overcounting

• However, we also have the two ‘a’s to consider!
• Fortunately, those equivalent permutations are simpler to count:

\[
onom a_1 \text{topoeia}_2 \\
onom a_2 \text{topoeia}_1
\]

• Key: for every one of these two (equivalent) permutations, we have 4! equivalent permutations because of the ‘o’s! (MULTIPLICATION RULE)
Complex Overcounting

• However, we also have the two ‘a’s to consider!
• Fortunately, those equivalent permutations are simpler to count:

\[
\begin{align*}
op\alpha_1 & \top\varepsilon\iota_2 \\
op\alpha_2 & \top\varepsilon\iota_1
\end{align*}
\]

• Key: **for every one** of these two (equivalent) permutations, we have 4! equivalent permutations because of the ‘o’s! **(MULTIPLICATION RULE)**
• Final answer:

\[
\text{#permutations} = \frac{12!}{4! \cdot 2!} = \frac{5 \cdot 6 \cdot \ldots \cdot 11 \cdot 12}{2} = 5 \cdot 2^2 \cdot \ldots \cdot 10 \cdot 11 = 9,979,200
\]
Important “Pedagogical” Note

• In the previous problem, we came up with the quantity

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\frac{12!}{4! \cdot 2!} = 9,979,200
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• How you should answer in an exam: \[
\frac{12!}{4! \cdot 2!}
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Important “Pedagogical” Note

• In the previous problem, we came up with the quantity

\[
\frac{12!}{4! \cdot 2!} = 9,979,200
\]

• How you should answer in an exam: \[\frac{12!}{4! \cdot 2!}\]

• Don’t perform computations, like 9,979,200
  • Helps you save time and us when grading 😊
For You!

• Consider the word “bookkeeper” (according to this website, the only unhyphenated word in English with three consecutive repeated letters)
For You!

• Consider the word “bookkeeper” (according to this website, the only unhyphenated word in English with three consecutive repeated letters)

• How many non-equivalent permutations of “bookkeeper” exist?
• Consider the word “bookkeeper” (according to this website, the only unhyphenated word in English with three consecutive repeated letters)

• How many non-equivalent permutations of “bookkeeper” exist?

$$\frac{10!}{2! \cdot 2! \cdot 3!}$$

Don’t forget the third ‘e’!
More Practice

• What about the #non-equivalent permutations for the word combinatorics
More Practice

• What about the #non-equivalent permutations for the word combinatorics

\[
\frac{13!}{2! \cdot 2! \cdot 2!} = \ldots
\]
General Template

- Total # permutations of a string $\sigma$ of letters of length $n$ where there are $n_a$ 'a's, $n_b$ 'b's, $n_c$ 'c's, ... $n_z$ 'z's

$$\frac{n!}{n_a! \times n_b! \times \cdots \times n_z!}$$
• Total # permutations of a string $\sigma$ of letters of length $n$ where there are $n_a$ 'a's, $n_b$ 'b's, $n_c$ 'c's, … $n_z$ 'z's

\[ \frac{n!}{n_a! \times n_b! \times \cdots \times n_z!} \]

• Claim: This formula is problematic when some letter (a, b, ..., z) is not contained in $\sigma$
General Template

• Total # permutations of a string $\sigma$ of letters of length $n$ where there are $n_a$ 'a's, $n_b$ 'b's, $n_c$ 'c's, ... $n_z$ 'z's.

$$\frac{n!}{n_a! \times n_b! \times \cdots \times n_z!}$$

• Claim: This formula is problematic when some letter (a, b, ..., z) is not contained in $\sigma$.

Remember: $0! = 1$
$r$-permutations

- Warning: permutations (as we’ve talked about them) are best presented with strings.
\( r \)-permutations

• Warning: permutations (as we’ve talked about them) are best presented with strings.

• \( r \)-permutations: Those are best presented with sets.
  • Note that \( r \in \mathbb{N} \)
  • So we can have 2-permutations, 3-permutations, etc
$r$-permutations: Example

- I have ten people.

- My goal: pick three people for a picture, where order of the people matters.
$r$-permutations: Example

- I have ten people.

- My goal: pick three people for a picture, where order of the people matters.
- Examples: shortest-to-tallest or tallest-to-shortest or something-in-between
r-permutations: Example

• I have ten people.

• My goal: pick three people for a picture, where order of the people matters.

• Examples: Jenny-Fred-Bob or Fred-Jenny-Bob or Fred-Bob-Jenny
$r$-permutations: Example

• I have ten people.

• My goal: pick three people for a picture, where order of the people matters.
• In how many ways can I pick these people?
I need three people for this photo. You guys figure out your order.
I need three people for this photo. You guys figure out your order.
I need three people for this photo. You guys figure out your order.

r-permutations: Example

10 ways to pick the first person...
I need three people for this photo. You guys figure out your order.

10 ways to pick the first person...

\(r\)-permutations: Example
I need three people for this photo. You guys figure out your order.

10 ways to pick the first person...
$r$-permutations: Example

I need three people for this photo. You guys figure out your order.

9 ways to pick the second person...
$r$-permutations: Example

I need three people for this photo. You guys figure out your order.

9 ways to pick the second person...
I need three people for this photo. You guys figure out your order.

9 ways to pick the second person...
I need three people for this photo. You guys figure out your order.

8 ways to pick the third person...
I need three people for this photo. You guys figure out your order.

8 ways to pick the third person...
I need three people for this photo. You guys figure out your order.

8 ways to pick the third person...
$r$-permutations: Example

I need three people for this photo. You guys figure out your order.

For a total of $10 \times 9 \times 8 = 720$ ways.
I need three people for this photo. You guys figure out your order.

For a total of $10 \times 9 \times 8 = 720$ ways.

Note: $10 \times 9 \times 8 = \frac{10!}{(10-3)!}$
Example on Books

• Clyde has the following books on his bookshelf
  • Epp, Rosen, Hughes, Bogart, Davis, Shaffer, Sellers, Scott
Example on Books

• Clyde has the following books on his bookshelf
  • Epp, Rosen, Hughes, Bogart, Davis, Shaffer, Sellers, Scott

• Jason wants to borrow any 5 of them and read them in the order he picks them in
Example on Books

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• Jason wants to borrow any 5 of them and read them in the order he picks them in.

• In how many ways can Jason get smart by reading those books?
Example on Books

• Clyde has the following books on his bookshelf
  • Epp, Rosen, Hughes, Bogart, Davis, Shaffer, Sellers, Scott

• Jason wants to borrow any 5 of them and read them in the order he picks them in.

• In how many ways can Jason get smart by reading those books?

\[
\frac{8!}{(8 - 5)!} = \frac{8!}{3!}
\]
• Let $n, r \in \mathbb{N}$ such that $0 \leq r \leq n$. The total ways in which we can select $r$ elements from a set of $n$ elements \textit{where order matters} is equal to:

$$P(n, r) = \frac{n!}{(n - r)!}$$
General Formula

• Let $n, r \in \mathbb{N}$ such that $0 \leq r \leq n$. The total ways in which we can select $r$ elements from a set of $n$ elements where order matters is equal to:

$$P(n, r) = \frac{n!}{(n - r)!}$$

“$P$” for permutation. This quantity is known as the $r$-permutations of a set with $n$ elements.
Pop Quizzes

1) \( P(n, 1) = \ldots \) \begin{array}{c}
0 \\
1 \\
n \\
n!
\end{array}
Pop Quizzes

1) \( P(n, 1) = \cdots \)

- Two ways to convince yourselves:
  - Formula: \( \frac{n!}{(n-1)!} = n \)
  - Semantics of \( r \)-permutations: In how many ways can I pick 1 element from a set of \( n \) elements? Clearly, I can pick any one of \( n \) elements, so \( n \) ways.
2) \( P(n, n) = \ldots \) 0 1 \( n \) \( n! \)
2) \( P(n, n) = \cdots \) 

- Formula: \( \frac{n!}{(n-n)!} = \frac{n!}{0!} \)
- Semantics: \( n! \) ways to pick all of the elements of a set and put them in order!
Pop Quizzes

3) $P(n, 0) = \ldots \begin{array}{c} 0 \\ 1 \\ n \\ n! \end{array}$
Pop Quizzes

3) \( P(n, 0) = \ldots \)

• Again, two ways to convince ourselves:
  • Formula: \( \frac{n!}{(n-0)!} = \frac{n!}{n!} = 1 \)
  • Semantics: Only one way to pick nothing: just pick nothing and leave!
Practice

1. How many MD license plates are possible to create?
Practice

1. How many MD license plates are possible to create? $26^2 \cdot 10^5$
Practice

1. How many MD license plates are possible to create? $26^2 \cdot 10^5$
2. How many ATM PINs are possible?
Practice

1. How many MD license plates are possible to create? $26^2 \cdot 10^5$
2. How many ATM PINs are possible? $10^4$
Practice

1. How many MD license plates are possible to create? $26^2 \cdot 10^5$
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3. How many words of length 10 can I construct from the English alphabet, where letters can be chosen:
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   a) With replacement (as in, I can reuse letters)
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   a) With replacement (as in, I can reuse letters) $26^{10}$
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1. How many MD license plates are possible to create? \(26^2 \cdot 10^5\)

2. How many ATM PINs are possible? \(10^4\)

3. How many words of length 10 can I construct from the English alphabet, where letters can be chosen:
   a) With replacement (as in, I can \textbf{reuse} letters) \(26^{10}\)
   b) Without replacement (as in, I \textbf{cannot reuse} letters) \(P(26, 10) = \frac{26!}{16!}\)
Practice

1. How many MD license plates are possible to create? $26^2 \cdot 10^5$

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3. How many words of length 10 can I construct from the English alphabet, where letters can be chosen:
   a) With replacement (as in, I can reuse letters) $26^{10}$
   b) Without replacement (as in, I cannot reuse letters) $P(26, 10) = \frac{26!}{16!}$

Remember these phrases!
Combinations (that “n choose r” stuff)

• Earlier, we discussed this example:

- I need three people for this photo. You guys figure out your order.

• Our goal was to pick three people for a picture, where order of the people mattered.
Combinations (that “n choose r” stuff)

• Earlier, we discussed this example:

• We now change this setup to forming a PhD defense committee (also 3 people).
• In this setup, does order matter?
Combinations (that “n choose r” stuff)
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We can make this selection in $P(10, 3)$ ways...
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We can make this selection in $P(10, 3)$ ways... but since order doesn't matter, we have 3! permutations of these people that are equivalent.
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Overcount 😞
In a precise way 😊
Combinations (that “n choose r” stuff)

We can make this selection in \( P(10, 3) \) ways… but since order doesn’t matter, we have \( 3! \) permutations of these people that are equivalent.

\[
\frac{P(10,3)}{3!} = \frac{10!}{7! \times 3!}
\]
Closer Analysis of Example

• Note that essentially we are asking you: Out of a set of 10 people, how many subsets of 3 people can I retrieve?
Notation

• The quantity

\[ P(10, 3) \]

\[ \frac{10!}{3!} \]

is the number of \textit{3-combinations} from a set of size 10, denoted thus:

\[ \binom{n}{3} \]

and pronounced “n choose 3”.

\[ \binom{n}{r} \] Notation

- Let \( n, r \in \mathbb{N} \) with \( 0 \leq r \leq n \)
- Given a set \( A \) of size \( n \), the total number of subsets of \( A \) of size \( r \) is:

\[ \binom{n}{r} = \frac{n!}{r!(n-r)!} \]
\((n \choose r)\) Notation

- Let \(n, r \in \mathbb{N}\) with \(0 \leq r \leq n\)

- Given a set \(A\) of size \(n\), the total number of subsets of \(A\) of size \(r\) is:

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- Pop quiz: \((\forall n, r \in \mathbb{N})[(0 \leq r \leq n) \Rightarrow \left(\binom{n}{r} \leq P(n, r)\right)]\)

- True
- False
\[ \binom{n}{r} \text{ Notation} \]

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- Pop quiz: \((\forall n, r \in \mathbb{N})[(0 \leq r \leq n) \Rightarrow \left( \binom{n}{r} \leq P(n, r) \right)]\)

Recall that
\[
\binom{n}{r} = \frac{P(n,r)}{r!} \text{ and } r! \geq 1
\]

True
False
Quiz
1. \( \binom{n}{1} = \)
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2. \( \binom{n}{n} = \)
1. \( \binom{n}{1} = n \)

2. \( \binom{n}{n} = 1 \) (Note how this differs from \( P(n, n) = n! \))
1. \( \binom{n}{1} = n \)

2. \( \binom{n}{n} = 1 \) (Note how this differs from \( P(n, n) = n! \))

3. \( \binom{n}{0} = \)
1. $\binom{n}{1} = n$

2. $\binom{n}{n} = 1$ (Note how this differs from $P(n, n) = n!$)

3. $\binom{n}{0} = 1$
STOP
RECORDING