**The Square Theorem**

**Definition** Let \( G \in \mathbb{N} \) and \( c \in N \). Let \( \text{COL}: [G] \times [G] \rightarrow [c] \).

1. A **mono** \( L \) is 3 points

\[(x, y), (x + d, y), (x, y + d)\]

that are all the same color \( (d \geq 1) \). This is an isosceles \( L \).

2. A **mono Square** is 4 points

\[(x, y), (x + d, y), (x, y + d), (x + d, y + d)\]

that are all the same color \( (d \geq 1) \). This is a square.
The Square Theorem

**Theorem** There exists $G$ such that for all $\text{COL} : [G] \times [G] \rightarrow [2]$ there exists a mono square.
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2. We will first prove *For all $c$ there exists $GG = GG(c)$ such that for all $\text{COL}: [GG] \times [GG] \rightarrow [c]$ there exists a mono $L$.*
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3. To prove The Square Theorem (about 2-coloring) we need to know that $GG(c)$ exists for a very large $c$.

4. More Colors: For all $c$ there exists $G = G(c)$ such that for all $\text{COL}: [G] \times [G] \to [c]$ there exists a mono square. Proof needs a larger $c'$ for $GG(c')$. 


The $L$ Theorem for $c = 2$

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Take the $[H] \times [H]$ grid and tile it with $3 \times 3$ tiles. View a 2-coloring of $[H] \times [H]$ as a $2^9$-coloring of the tiles.
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This is very typical of VDW-Ramsey Theory: a 2-coloring of BLAH is viewed as a $\times$-coloring of a different object where $\times$ is quite large.
Why This Size Tile?

Any 2-coloring of the $3 \times 3$ tile will have two of the same color in the first column and hence an almost $L$. 
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Goto Zoom-White Board.
Make $H$ Big Enough To Get Two Tiles Same Color

Take $H = 3(2^9 + 1)$. 
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Look at the first column of tiles. Two are the same color.
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First take $4 \times 4$-tiles.
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Make $H$ Big Enough To Get Two Tiles Same Color

Take $H = 4(3^{16} + 1)$. 
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View $[H] \times [H]$ grid of points as $[3^{16} + 1] \times [3^{16} + 1]$ grid of tiles.
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Take Much Bigger Tiles

Take Tile so big that any 3-coloring of it has two different colored almost-L’s converging to the same point.
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*We won't prove this but I am sure any of you could prove it given what we have done so far. Would be messy.*

*Easier to prove it from the Hales-Jewitt Theorem, which we won't be doing.*
Full L Theorem

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**Theorem** There exists $G$ such that for all $\text{COL} : [G] \times [G] \rightarrow [2]$ there exists a mono square.

Proof

$G$ will be $G(2) \times G(2)$.

Tile the $G \times G$ plane with $G(2) \times G(2)$ tiles.

View the 2-coloring of $G \times G$ as a 2-coloring of the tiles.

For any 2-coloring of $G \times G$:

- Every tile has a mono tile.
- There is a mono tile of tiles.

Go to Zoom Whiteboard for rest of proof.
The Square Theorem

**Theorem** There exists $G$ such that for all $\text{COL} : [G] \times [G] \rightarrow [2]$ there exists a mono square.

**Proof** $G$ will be $GG(2)GG(2^{GG(2)^2})$. 

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