Sets of Functions that are Uncountable

Exposition by William Gasarch

May 4, 2022
The set of all Functions from $\mathbb{N}$ to $\mathbb{N}$

**Thm** The set of all functions from $\mathbb{N}$ to $\mathbb{N}$ is uncountable.
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**Pf**

Assume, BWOC, that set of functions \( \mathbb{N} \) to \( \mathbb{N} \) is countable.

Then we can list them out \( f_1, f_2, \ldots \).

Consider the function \( F(x) = f_x(x) + 1 \).

\( F \) cannot be \( f_1 \) since \( F(1) \neq f_1(1) \).

\( F \) cannot be \( f_{83} \) since \( F(83) \neq f_{83}(83) \).

For all \( i \), \( F \) cannot be \( f_i \) since \( F(i) \neq f_i(i) \).

So \( F \) is NOT on the list and IS from \( \mathbb{N} \) to \( \mathbb{N} \), contradiction.
Thm The set of all functions from \( \mathbb{N} \) to \( \mathbb{N} \) is uncountable.

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Assume, BWOC, that set of functions $\mathbb{N}$ to $\mathbb{N}$ is countable. Then we can list them out $f_1, f_2, \ldots$. Consider the function

$$F(x) = f_x(x) + 1.$$
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$$F(x) = f_x(x) + 1.$$  

$F$ cannot be $f_1$ since $F(1) \neq f_1(1)$. 
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\( F \) cannot be \( f_1 \) since \( F(1) \neq f_1(1) \).

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$F$ cannot be $f_1$ since $F(1) \neq f_1(1)$.
$F$ cannot be $f_{83}$ since $F(83) \neq f_{83}(83)$.
For all $i$, $F$ cannot be $f_i$ since $F(i) \neq f_i(i)$.
So $F$ is NOT on the list and IS from $\mathbb{N}$ to $\mathbb{N}$, contradiction.
Key to Last Proof

We had to make sure that the final object we produced was a function from $\mathbb{N}$ to $\mathbb{N}$. For future proofs you must check both properties.
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We had to make sure that the final object we produced was

- Not in the list
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For future proofs you must check both properties.
The set of all Functions from $\mathbb{N}$ to EVENS

**Thm** The set of all functions from $\mathbb{N}$ to EVENS is uncountable.

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*Proof* Assume, BWOC, the set of functions from $\mathbb{N}$ to EVENS is countable. Then we can list them out as $f_1, f_2, \ldots$. Consider the function $F(x) = f_x(x) + 1$. CAN'T USE THIS! $F$ is not from $\mathbb{N}$ to EVENS. What to do? Let $F(x) = f_x(x) + 2$. Then $F$ is a function from $\mathbb{N}$ to EVENS, contradicting the assumption that the set is countable. Therefore, the set of all functions from $\mathbb{N}$ to EVENS is uncountable.
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CAN'T USE THIS! $F$ is not from $\mathbb{N}$ to $\text{EVENS}$. What to do?

$$F(x) = f_x(x) + 2.$$
Thm  The set of all functions from $\mathbb{N}$ to SQUARES is uncountable.
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**Pf**
Assume, BWOC, set of all functions $\mathbb{N}$ to SQUARES is countable.

Then can list them out $f_1, f_2, \ldots$.
We want to define $F$ so that $F(x) \neq f_x(x)$ AND $F(x)$ is a square.

$$F(x) = (f_x(x) + 1)^2$$

Make sure this is not $f_x(x)$:

$$(f_x(x) + 1)^2 \neq f_x(x)^2 + 2f_x(x) + 1$$

**IF** $f_x(x)^2 + 2f_x(x) + 1 = f_x(x)$ then $f_x(x)^2 + f_x(x) + 1 = 0$

Only has complex solutions, so can't happen.
The set of all Functions from \( \mathbb{N} \) to SQUARES

**Thm** The set of all functions from \( \mathbb{N} \) to SQUARES is uncountable.

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(f_x(x) + 1)^2 = f_x(x)^2 + 2f_x(x) + 1
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Make sure this is not \( f_x(x) \):
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(f_x(x) + 1)^2 = f_x(x)^2 + 2f_x(x) + 1
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IF \( f_x(x)^2 + 2f_x(x) + 1 = f_x(x) \) then \( f_x(x)^2 + f_x(x) + 1 = 0 \)
Only has complex solutions, so can’t happen.
The set of constant functions is countable since here they are:

\[ f_1(x) = 1 \]
\[ f_2(x) = 2 \]

etc.
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What goes wrong if we try to prove that the set of constant functions is uncountable?
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etc.

What goes wrong if we try to prove that the set of constant functions is uncountable?

\[ F(x) = f_x(x) + 1. \]
The set of all Constant Functions With Domain \( \mathbb{N} \)

The set of constant functions is countable since here they are:
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\begin{align*}
  f_1(x) &= 1 \\
  f_2(x) &= 2 \\
  \text{etc.}
\end{align*}
\]

What goes wrong if we try to prove that the set of constant functions is uncountable?

\[
F(x) = f_x(x) + 1.
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1) \( F \) is NOT on the list. Good!
The set of all Constant Functions With Domain $\mathbb{N}$

The set of constant functions is countable since here they are:  
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f_1(x) = 1 \\
f_2(x) = 2 \\
etc.
\]
What goes wrong if we try to prove that the set of constant functions is uncountable?

\[
F(x) = f_x(x) + 1.
\]

1) $F$ is NOT on the list. Good!
2) But $F$ is not constant. So proof fails.
Intuition

\( \mathbb{Q}, \mathbb{N}, \mathbb{Z}, \mathbb{N} \times \mathbb{N} : \)

Every element of \( \mathbb{Q} \) can be described with finite number of bits.

Every element of \( \mathbb{N} \) can be described with finite number of bits.

\( \mathbb{R} \), \( \{ f : \mathbb{N} \rightarrow \mathbb{N} \} \),

Every element of \( \mathbb{R} \) requires an infinite number of bits to represent.

Not quite right: Some elements of \( \mathbb{R} \) are easy to describe, e.g., \(-3, 5, 12\).

But most elements of \( \mathbb{R} \) require an infinite number of bits.

Rule of Thumb

Let \( A \) be an infinite set.

▶ If every element of \( A \) can be represented with a finite number of bits then \( A \) is countable.

▶ If an infinite number of elements of \( A \) require an infinite number of bits to be represented, then \( A \) is NOT countable.
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$\mathbb{Q}, \mathbb{N}, \mathbb{Z}, \mathbb{N} \times \mathbb{N}$:

Every element of $\mathbb{Q}$ can be described with FINITE number of bits. Every element of $\mathbb{N}$ can be described with FINITE number of bits.
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$\mathbb{R}$, $\{f : \mathbb{N} \to \mathbb{N}\}$,
Every element of $\mathbb{R}$ requires an INFINITE number of bits to represent.
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Every element of \( \mathbb{R} \) requires an \textbf{INFINITE} number of bits to represent.
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But \textbf{most} elements of \( \mathbb{R} \) require an infinite number of bits.

**Rule of Thumb**

Let \( A \) be an infinite set.

- If every element of \( A \) can be represented with a finite number of bits then \( A \) is countable
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