Nim Games

250H
How to Play

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  - You can have any number of piles and any amount of objects in each pile
- Each player must remove at least 1 object and may remove any number of objects as long as they all come from the same pile
- Depending on the version: the goal of the game is either to
  - Avoid taking the last object
  - To take the last object
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Consider a 2 pile game of Nim where you win if you don’t have to pick up the last stone. Prove if both piles of stones have n stones each and it’s the first player’s turn, the second player can always win.

**Base Case:** If both piles have 0 stones in them, the first player loses
Consider a 2 pile game of Nim where you win if you don’t have to pick up the last stone. Prove if both piles of stones have \( n \) stones each and it’s the first player’s turn, the second player can always win.

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**Inductive Hypothesis:** Assume that for some \( n \geq 0 \) and \( 0 \leq i < n \). If both piles have \( i \) number of stones and it’s the first player’s turn, the second player can win.
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**Inductive Step:** Consider a game of Nim in which there are two piles of stones, A and B, with n stones in each. Without loss of generality, let A be the pile that the first player chooses to remove stones from.
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The first player must remove \( k \) stones from pile A such that \( 1 \leq k \leq n \). So, we have \( n - k \) stones in pile A and \( n \) stones in pile B.
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**Inductive Step:** Consider a game of Nim in which there are two piles of stones, A and B, with $n$ stones in each. Without loss of generality, let A be the pile that the first player chooses to remove stones from.

The first player must remove $k$ stones from pile A such that $1 \leq k \leq n$. So, we have $n - k$ stones in pile A and $n$ stones in pile B.

If the second player removes $k$ stones from pile B, both piles have $n - k$ stones in each.

By the induction hypothesis, the second player can now win this game because there are two piles with $n-k$ stones in each.
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- Example:
  - Pile 1 has 2 objects
  - Pile 2 has 3 objects
  - \[ 10 + 11 = 01 \]
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    ▪ The Nim sum of the number of objects after your move will not be equal to 0
  ○ Suppose it's your turn and the Nim sum of the number of objects in the pile is not equal to 0
    ▪ Then there is a move which ensures that the Nim sum of the number of objects in the pile after your move is equal to 0
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- **Player 1’s strategy:** if possible always make a move that reduces the Nim sum after your move to 0
- This would then mean that whatever player 2 does next, the move would turn the next Nim sum into a number that's not 0
- Player 1 wins IFF there is a move he can make that puts the game into a Player 2 win position
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  - The number N+1 is good if at least one bad number can be reached by subtracting a positive square
- The winning strategy of the game: Try to pass on a bad number to your opponent
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  - If there are only 1, 2, or 3 objects left on your turn, you take all of them
  - If you have to move when there are 4 objects you will always lose
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  - If there are 5, 6, or 7 objects, you can win by taking just enough to leave 4 objects.
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  - The winning strategy of the game: At the end of your turn, make it so that your opponent is taking from a multiple of 4 objects
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  - If there are 6, you can win by taking 4 objects.
  - If you have to move when there are 7 objects you will always lose.
  - **The winning strategy of the game**: At the end of your turn, make it so that your opponent is taking from a pile that is equivalent to 2 or 0 mod 7.
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  - The classification of positions into hot and cold can be looked at recursively:
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    - Any position from which a cold position can be reached in a single move is a hot position
    - If every move leads to a hot position, then a position is cold.