Nim Games

250H

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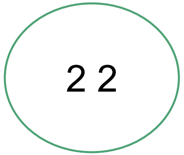
  - You can have any number of piles and any amount of objects in each pile
- Each player must remove at least 1 object and may remove any number of objects as long as they all come from the same pile
- Depending on the version: the goal of the game is either to
  - Avoid taking the last object
  - To take the last object

We can use game trees to look at all possible games (the players are playing perfectly here)



Player 1’s turn

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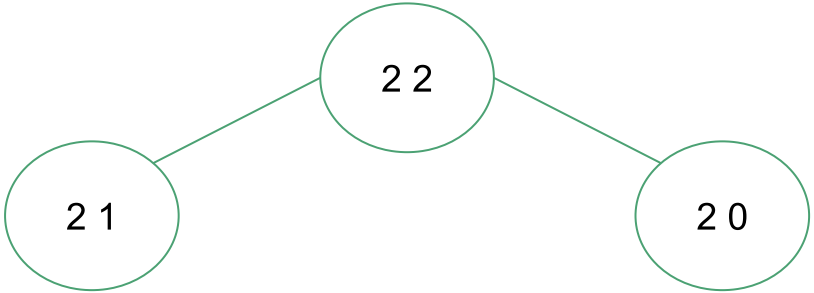

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Player 2's turn

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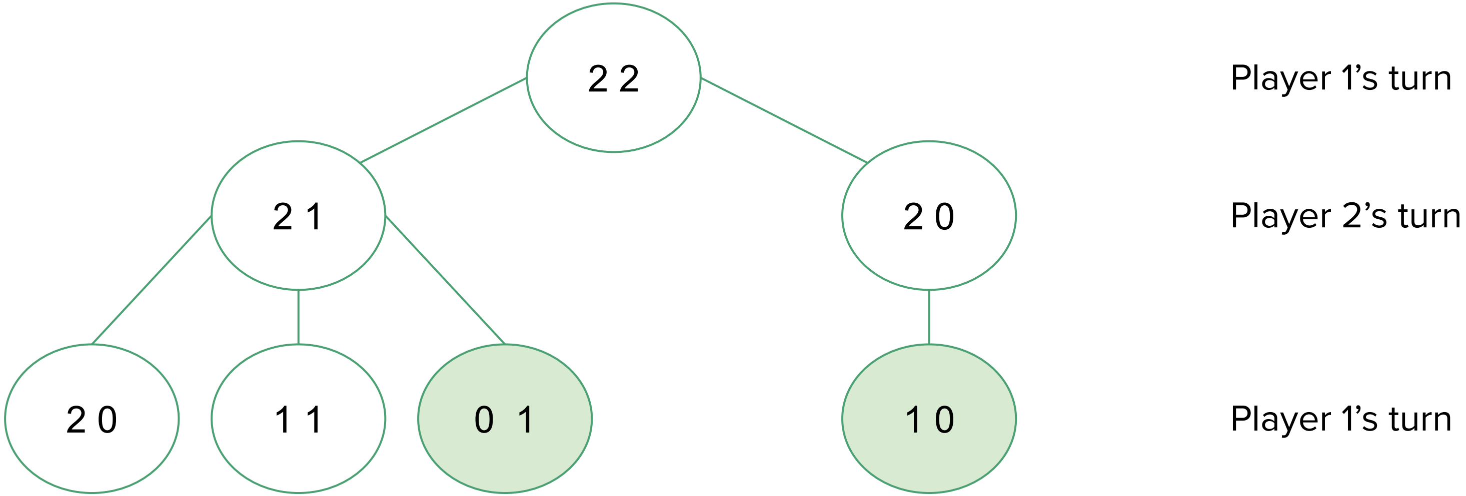

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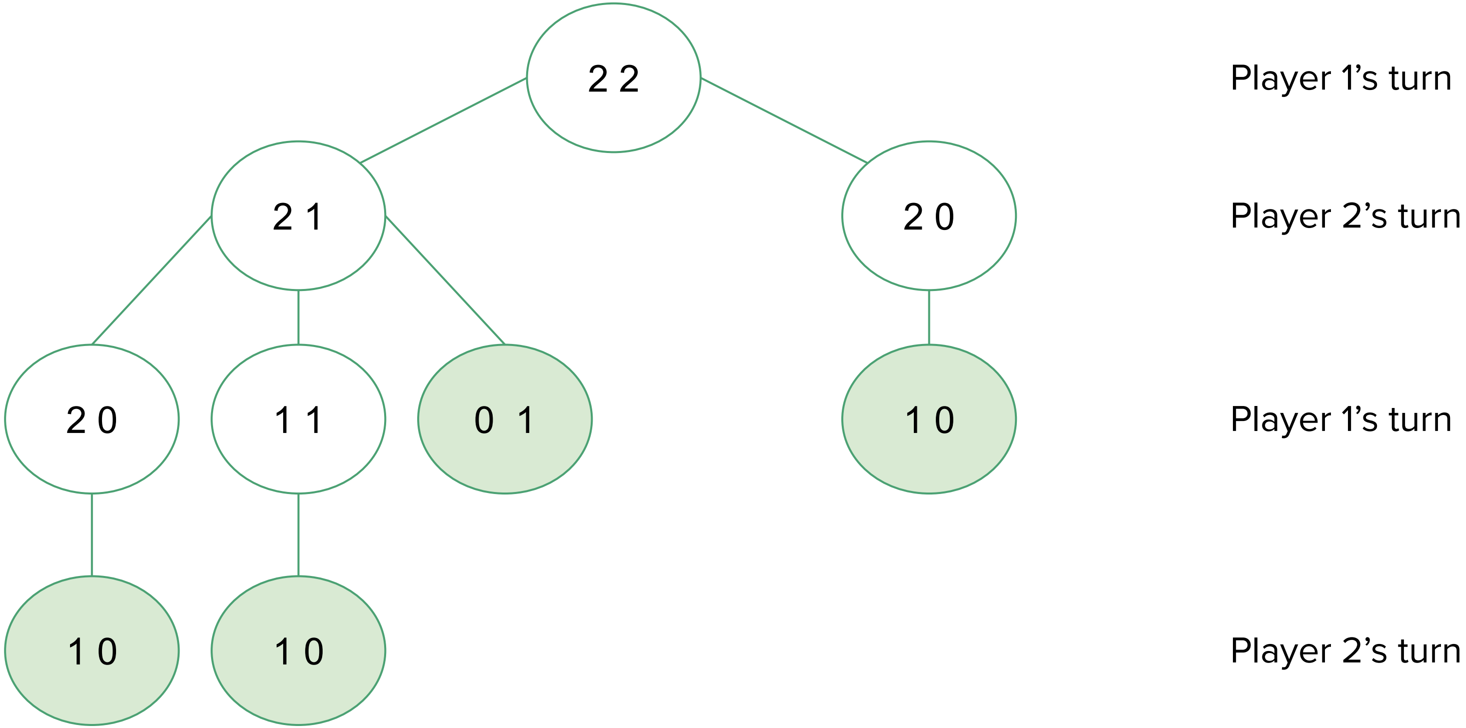

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Inductive Step: Consider a game of Nim in which there are two piles of stones, A and B, with n stones in each. Without loss of generality, let A be the pile that the first player chooses to remove stones from.

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By the induction hypothesis, the second player can now win this game because there are two piles with $n-k$ stones in each.

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- Example:
    - Pile 1 has 2 objects
    - Pile 2 has 3 objects
    - $$\begin{array}{r} 10 \\ +\,11 \\ \hline 01 \end{array}$$

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  - Suppose it's your turn and the Nim sum of the number of objects in the pile is not equal to 0
    - Then there is a move which ensures that the Nim sum of the number of objects in the pile after your move is equal to 0

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- Player 1 wins IFF there is a move he can make that puts the game into a Player 2 win position

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    - The winning strategy of the game: Try to pass on a bad number to your opponent

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    - If every move leads to a hot position, then a position is cold.