The Emptier-Filler Game

Consider the following games played between EMPTIER and FILLER. We denote EMPTIER by E and FILLER by F.

1. FILLER fills a box with a finite number of balls, each with a natural number of his choice on it.

2. In every round E takes a ball of his choice from the box. F then counters by replacing the ball with a finite number of other balls, each with a smaller number. (For example E takes a ball labeled 1000, then F replaces it with 999999999 balls labeled 999 and 888876234012 balls labeled 8.)

If the box is ever empty then E wins. If the box is always nonempty (i.e., the game goes on forever) then F wins.

**QUESTION:** Is there a strategy that E can play so that he will ALWAYS win? Is there a strategy that F can play so that he can ALWAYS win?

**VARIANTS:**

1. The balls are labeled with integers.

2. The balls are labeled with rationals that are $\geq 0$.

3. The balls are labeled with ordered pairs of naturals and the ordering is

   $$(0, 0) < (0, 1) < (0, 2) < \cdots (1, 0) < (1, 1) < (1, 2) < \cdots (2, 0) < (2, 1) < (2, 2) < \cdots$$

   (That is, if E removes $(i, j)$ then F can put in as many balls as he wants that are labeled with ordered pairs that are LESS THAN $(i, j)$ in this ordering.)

4. The balls are labeled with ordered triples of natural numbers. Let the ordering be $(a, b, c) < (d, e, f)$ if either (1) $a < d$ or (2) $a = d$ and $b < e$, or (3) $a = d$ and $b = e$ but $c < f$.

**QUESTION:** Let $X$ be a set and $\leq$ be an ordering on it. Let the $(X, \leq)$-game be the game as above where we label the balls with elements from $A$.

In the following sentence fill in the ???

“E can win the $(X, \leq)$-game if and only if $(X, \leq)$ has property ???”