What is a set?

• A set is a collection of distinct objects.
• We use the notation $x \in S$ to say that $S$ contains $x$.
• We’d like to know if $x \in S$ fast!
• Unless explicitly specified otherwise, sets are unordered.
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- Doubly Linked List
- Binary Tree
- Stack
- Something else (what?)
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**Hash table!**
Elementary number sets

- **ℕ**: the **natural** numbers
  - \( \mathbb{N} = \{0, 1, 2, 3, \ldots \} \). In our class, \( 0 \in \mathbb{N} \)!

- **ℤ**: the **integers**
  - \( \mathbb{Z} = \{ \ldots -3, -2, -1, 0, 1, 2, 3, \ldots \} \)

- **ℚ**: the **rationals**
  - \( \mathbb{Q} = \{ \frac{a}{b}, (a \in \mathbb{Z}) \land (b \in \mathbb{Z}) \land (b \neq 0) \} \)
  - Any number that can be written as a ratio of integers!

- **ℝ**: the **reals**
  - This will typically be our “upper limit” in 250.
  - That is, we don’t usually care about \( \mathbb{C} \), the set of **complex** numbers
Fill those in!

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Not even a real number!
Venn Diagrams
• *U* is the *Universal Domain*: a set that we imagine holds every *conceivable* element.

• When talking about sets of numbers, *U* is usually ℝ, the reals.
“There exists” ($\exists$)

• The symbol $\exists$ (LaTeX: \exists) is read “There exists”.
• Examples:
  • $(\exists x \in \mathbb{R}) [8x = 1]$
“There exists” ($\exists$)

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• Examples:
  • $(\exists x \in \mathbb{R}) [8x = 1]$ True
“There exists” (∃)

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• Examples:
  • \((\exists x \in \mathbb{R}) [8x = 1]\) True
  • \((\exists n \in \mathbb{Z}) [n^2 = -1]\)
“There exists” (∃)

• The symbol $\exists$ (LaTeX: \exists) is read “There exists”.
• Examples:
  • $(\exists x \in \mathbb{R}) [8x = 1]$ True
  • $(\exists n \in \mathbb{Z})[n^2 = -1]$ False
“There exists” (∃)

• The symbol ∃ (LaTeX: \exists) is read “There exists”.
• Examples:
  • (∃x ∈ ℝ) [8x = 1] True
  • (∃n ∈ ℤ)[n² = −1] False
• Is there a domain D where (∃n ∈ D)[n² = −1] is true?
  Yes  No  Something else
“There exists” ($\exists$)

- The symbol $\exists$ (LaTeX: \exists) is read “There exists”.
- Examples:
  - $(\exists x \in \mathbb{R}) [8x = 1]$ True
  - $(\exists n \in \mathbb{Z})[n^2 = -1]$ False
- Is there a domain $D$ where $(\exists n \in D)[n^2 = -1]$ is true?
  
  The complex numbers $\mathbb{C}$
  
  Yes  No  Something else
“For all” (\(\forall\))

- The symbol \(\forall\) (LaTeX: \(\forall\)) is read “for all”.
- Examples:
  - \((\forall x \in \mathbb{N}) [((x > 2) \land (x \text{ is prime})) \Rightarrow (x \text{ is odd})]\)
“For all” (\(\forall\))

- The symbol \(\forall\) (LaTeX: `\forall`) is read “for all”.
- Examples:
  - \((\forall x \in \mathbb{N}) [((x > 2) \land (x \text{ is prime})) \Rightarrow (x \text{ is odd})]\)
  - True
“For all” (∀)

• The symbol ∀ is read “for all”.
• Examples:
  • (∀x ∈ ℕ) [((x > 2) ∧ (x is prime)) ⇒ (x is odd)]
    True
  • (∀n ∈ ℤ) [n^2 ≥ 0]
“For all” \((\forall)\)

• The symbol \(\forall\) is read “for all”.

• Examples:
  • \((\forall x \in \mathbb{N}) [((x > 2) \land (x \text{ is prime})) \implies (x \text{ is odd})]\)
    True
  • \((\forall n \in \mathbb{Z}) [n^2 \geq 0]\) True
“For all” ($\forall$)

• Let $D$ be the set of all students in this class who are over 8 feet tall.
• ($\forall x \in D)[x \text{ has perfect attendance so far!]
“For all” (∀)

• Let D be the set of all students in this class who are over 8 feet tall.
• (∀x ∈ D)[x has perfect attendance so far!]

If disagree, need to find x ∈ D who missed a class
• Called vacuously true!
Nesting quantifiers

• \((\exists x \in \mathbb{N})(\exists y \in \mathbb{N})[x + 2y = 3x + y = 4]\)
Nesting quantifiers

• \((\exists x \in \mathbb{N})(\exists y \in \mathbb{N})[x + 2y = 3x + y = 4]\) False
Nesting quantifiers

• $(\exists x \in \mathbb{N})(\exists y \in \mathbb{N})[x + 2y = 3x + y = 4]$ False
• $(\exists x \in \mathbb{Q})(\exists y \in \mathbb{Q})[x + 2y = 3x + y = 4]$
Nesting quantifiers

• \((\exists x \in \mathbb{N})(\exists y \in \mathbb{N})[x + 2y = 3x + y = 4]\) False
• \((\exists x \in \mathbb{Q})(\exists y \in \mathbb{Q})[x + 2y = 3x + y = 4]\)
  True, \(x = \frac{4}{5}, y = \frac{8}{5}\)
Nesting quantifiers

• $(\exists x \in \mathbb{N})(\exists y \in \mathbb{N})[x + 2y = 3x + y = 4]$ False
• $(\exists x \in \mathbb{Q})(\exists y \in \mathbb{Q})[x + 2y = 3x + y = 4]$
  True, $x = \frac{4}{5}, y = \frac{8}{5}$

• Common abbreviation: $(\exists x, y \in D)[...]$
• Generally: $(\exists x_1, x_2, ..., x_n \in D)[...]$
Alternating nested quantifiers

• Notice the differences between the following:
  • \((\forall x \in \mathbb{N})(\exists y \in \mathbb{N})[x < y]\)
  • \((\exists x \in \mathbb{N})(\forall y \in \mathbb{N})[x < y]\)
Alternating nested quantifiers

• Notice the differences between the following:
  • $(\forall x \in \mathbb{N})(\exists y \in \mathbb{N})[x < y]$ True ($\mathbb{N}$ unbounded from above)
  • $(\exists x \in \mathbb{N})(\forall y \in \mathbb{N})[x < y]$
Alternating nested quantifiers

• Notice the differences between the following:
  • $(\forall x \in \mathbb{N})(\exists y \in \mathbb{N})[x < y]$ True ($\mathbb{N}$ unbounded from above)
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Alternating nested quantifiers

• Notice the differences between the following:
  • $(\forall x \in \mathbb{N})(\exists y \in \mathbb{N})[x < y]$ True ($\mathbb{N}$ unbounded from above)
  • $(\exists x \in \mathbb{N})(\forall y \in \mathbb{N})[x < y]$ False ($\mathbb{N}$ bounded from below)

• **WHEN QUANTIFIERS ARE DIFFERENT, THEIR ORDER MATTERS!!!!!!**
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</thead>
<tbody>
<tr>
<td>$(\exists n \in \mathbb{N})[n + n = 0]$</td>
<td>![Green Circle]</td>
<td></td>
</tr>
<tr>
<td>$(\exists n \in \mathbb{N})[n + n = 1]$</td>
<td></td>
<td>![Circle]</td>
</tr>
<tr>
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| $(\forall x, y \in \mathbb{N})(\exists z \in \mathbb{Q})[(x < y) \Rightarrow (x < z < y)]$ | ![Circle] |      |}

$n = 0$
Fill this in!

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$n = 0$

$2n = 1 \Rightarrow n = \frac{1}{2} \notin \mathbb{N}$
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$n = 0$

$2n = 1 \Rightarrow n = \frac{1}{2} \notin \mathbb{N}$

Similarly, $\frac{1}{2} \notin \mathbb{Z}$
Fill this in!

\[
\begin{array}{|c|c|c|}
\hline
\text{Statement} & \text{True} & \text{False} \\
\hline
(\exists n \in \mathbb{N})[n + n = 0] & \text{✓} & \text{✗} \\
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\hline
\end{array}
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\[n = 0\]
\[2n = 1 \Rightarrow n = \frac{1}{2} \notin \mathbb{N}\]

Similarly, \(\frac{1}{2} \notin \mathbb{Z}\)
x = 0, y = 1 or x = −1, y = 2, or...
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\(x^2 + x + 1 = 0\) has no real solutions
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Think of graph of $f(x) = x^2$
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- E.g: arithmetic mean
Finding domains

• Give infinite sets $D$ such that $(\forall x \in D)(\exists y \in D)[x < y]$
  1. Is true
Finding domains

• Give infinite sets $D$ such that $(\forall x \in D)(\exists y \in D)[x < y]$
  1. Is true ($D = \mathbb{N}$, select $y = x + 1$)
Finding domains

• Give infinite sets $D$ such that $(\forall x \in D)(\exists y \in D)[x < y]$
  1. Is true ($D = \mathbb{N}$, select $y = x + 1$)
  2. Is false
Finding domains

• Give infinite sets $D$ such that $(\forall x \in D)(\exists y \in D)[x < y]$
  1. Is true ($D = \mathbb{N}$, select $y = x + 1$)
  2. Is false ($D = \mathbb{Z}_{\leq 0}$, counter-example is 0)
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  1. Is true ($D = \mathbb{N}$, select $y = x + 1$)
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• Do the same thing for

$$(\forall x \in D)[x \leq 1] \land (\forall x \in D)(\exists y \in D)[x < y]$$
Finding domains

• Give infinite sets $D$ such that $(\forall x \in D)(\exists y \in D)[x < y]$
  1. Is true ($D = \mathbb{N}$, select $y = x + 1$)
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• Do the same thing for

$$(\forall x \in D)[x \leq 1] \land (\forall x \in D)(\exists y \in D)[x < y]$$

  1. True for $D = (-\infty, 1)$
Finding domains

• Give infinite sets $D$ such that $(\forall x \in D)(\exists y \in D)[x < y]$
  1. Is true ($D = \mathbb{N}$, select $y = x + 1$)
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• Do the same thing for $\forall x \in D \left[ x \leq 1 \right] \land (\forall x \in D)(\exists y \in D)[x < y]$

  1. True for $D = (-\infty, 1)$
  2. False for $D = (-\infty, 1]$ (!)
We say that $A$ is a subset of $B$ ($A \subseteq B$) iff

$$\forall x \in A \left[ x \in B \right]$$

$$\iff$$

$$\forall x \in U \left[ (x \in A) \Rightarrow (x \in B) \right]$$
Superset and proper subset/superset

• We say that $B$ is a **superset** of $A$ ($B \supseteq A$) iff $A \subseteq B$.
• We say that $A$ is a **proper subset** of $B$ ($A \subset B$) iff $(A \subseteq B) \land (A \neq B) \land (A \neq \emptyset)$.
• We say that $B$ is a **proper superset** of $A$ ($B \supset A$) iff $A \subset B$
Union

\[ A \cup B = \{ (x \in A) \lor (x \in B) \} \]
Union

\[ A \cup B = \{(x \in A) \vee (x \in B)\} \]

Connection between union and logical disjunction!
Intersection

\[ A \cap B = \{(x \in A) \land (x \in B)\} \]
Absolute complement

\[ A^c = \{ (x \notin A) \} = \{ (x \in U) \land (\neg (x \in A)) \} \]
Absolute complement

\[ A^c = \{ x \notin A \} = \{ (x \in U) \land (\neg (x \in A)) \} \]

Connection between absolute complement and logical negation!
Absolute complement

$$A^c = \{x \notin A\} = \{(x \in U) \land (\sim (x \in A))\}$$

Some use $A'$. They are Wrong, we are right.

Connection between absolute complement and logical negation!
Relative Complement

\[ A - B = \{(x \in A) \land (x \notin B)\} \]
Relative Complement

\[ A - B = \{ (x \in A) \land (x \notin B) \} \]

Some use \( A \setminus B \). They are wrong, we are right!
Careful about membership and subset!

• Be careful to distinguish between **members** of a set and **subsets** of a set...

  True  

  False
Careful about membership and subset!

• Be careful to distinguish between members of a set and subsets of a set...

1. $1 \in \{-2, 0, 1, 3\}$
Careful about membership and subset!

• Be careful to distinguish between **members** of a set and **subsets** of a set...

1. $1 \in \{-2, 0, 1, 3\}$ **True**
2. $1 \in \{-2, 0, \{1\}, 3\}$ **False**
Careful about membership and subset!

• Be careful to distinguish between members of a set and subsets of a set...

1. $1 \in \{-2, 0, 1, 3\}$ \text{T}
2. $1 \in \{-2, 0, \{1\}, 3\}$ \text{F}
3. $1 \subseteq \{-2, 0, \{1\}, 3\}$
Careful about membership and subset!

• Be careful to distinguish between **members** of a set and **subsets** of a set...

1. $1 \in \{-2, 0, 1, 3\}$ **T**
2. $1 \in \{-2, 0, \{1\}, 3\}$ **F**
3. $1 \subseteq \{-2, 0, \{1\}, 3\}$ **F**, in fact, not even mathematically correct syntax
4. $\{1\} \subseteq \{-2, 0, \{1\}, 3\}$
Careful about membership and subset!

• Be careful to distinguish between members of a set and subsets of a set...

1. $1 \in \{-2, 0, 1, 3\}$ True
2. $1 \in \{-2, 0, \{1\}, 3\}$ False
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5. $\{1\} \in \{-2, 0, \{1\}, 3\}$
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• Be careful to distinguish between **members** of a set and **subsets** of a set...

1. \(1 \in \{-2, 0, 1, 3\}\) \(T\)
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3. \(1 \subseteq \{-2, 0, \{1\}, 3\}\) \(F\), in fact, not even mathematically correct syntax
4. \(\{1\} \subseteq \{-2, 0, \{1\}, 3\}\) \(F\)
5. \(\{1\} \in \{-2, 0, \{1\}, 3\}\) \(T\)
6. \(\{1\} \subseteq \{-2, 0, 1, 3\}\)
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The empty set ($\emptyset, \{ \}$)

• The empty set, denoted either $\emptyset$ or $\{ \}$, is the **unique** set with no elements.
  • Uniqueness can be proven, through a proof by contradiction!
The empty set \((\emptyset, \{\})\)

- The empty set, denoted either \(\emptyset\) or \(\{\}\), is the **unique** set with **no elements**.
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[True] [False]
The empty set ($\emptyset, \{ \}$)

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1. $\emptyset \subseteq \mathbb{N}$
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The powerset

• Given a set $A$, the powerset $\mathcal{P}(A)$ is the set of all subsets of $A$.

  • $\mathcal{P}([0, 1]) = \{\emptyset, \{0\}, \{1\}, \{0, 1\}\}$
  • $\mathcal{P}([0, 1, 2]) = \{\emptyset, \{0\}, \{1\}, \{2\}, \{0, 1\}, \{1, 2\}, \{0, 2\}, \{0, 1, 2\}\}$
  • Evens, Odds, Primes, Squares
    • And lots more...
Facts about the powerset

• The following are **facts** about the powerset:
  • Since \( \emptyset \subseteq A \) for all sets \( A \), \( \emptyset \in \mathcal{P}(A) \) for all sets \( A \)
  • Since \( A \subseteq A \) for all sets \( A \), \( A \in \mathcal{P}(A) \) for all sets \( A \)
Powerset quizzing

• Let $A = \{1, 2, ..., n\}$
• Then, $|P(A)|$

$\approx n \cdot \log n \quad = n^2 \quad = 2^n \quad = n!$
Let $A = \{1, 2, \ldots, n\}$
Then, $|P(A)|$

\[
\approx n \cdot \log n = n^2 = 2^n = n!
\]
Powerset quizzing

• $P(\{1\}) =$
Powerset quizzing

• $P(\{1\}) = \{\emptyset, \{1\}\}$
• $P(P(\{1\})) =$
Powerset quizzing

• $P(\{1\}) = \{\emptyset, \{1\}\}$
• $P(P(\{1\})) = \{\emptyset, \{\emptyset\}, \{\{1\}\}, \{\emptyset, \{1\}\}\}$
• $P(\emptyset) =$
Powerset quizzing

• \( P(\{1\}) = \{\emptyset, \{1\}\} \)
• \( P(P(\{1\})) = \{\emptyset, \{\emptyset\}, \{\{1\}\}, \{\emptyset, \{1\}\}\} \)
• \( P(\emptyset) = \{\emptyset\} \)
• \( P(\{\emptyset\}) = \)
Powerset quizzing

• $P(\{1\}) = \{\emptyset, \{1\}\}$
• $P(P(\{1\})) = \{\emptyset, \emptyset, \{\emptyset\}, \{\{1\}\}, \{\emptyset, \{1\}\}\}$
• $P(\emptyset) = \{\emptyset\}$
• $P(\{\emptyset\}) = \{\emptyset, \{\emptyset\}\}$
STOP
RECORDING